Wolfgang Lenzen

THE LOGIC OF 'IS PROBABLE'

Probability theory has mainly been concerned with the study either of quantitative (metric) or of comparative probability concepts. Although the latter are often referred to as 'qualitative probability', genuinely qualitative (classificatory) probability concepts do not seem to have received much attention. The only exception known to me is Gärdenfors's [1] where the logic of 'is maximally probable' is investigated. This paper deals with a less exclusive modality, namely with the concept of being probable in the sense of being more probable than not.

If $\geq$ denotes the two-place sentential operator 'is at least as probable as', then the logic of comparative probability may be obtained by adding to the propositional calculus (PC) the following set of axioms and deduction rules (cf. [1]), where

$(p > q) =_{\mathrm{df}} \neg(q \geq p)$;

$(p \sim q) =_{\mathrm{df}} (p \geq q) \wedge (q \geq p)$;

$\mathbf{T} =_{\mathrm{df}} (p \vee \neg p)$;

$\bot =_{\mathrm{df}} \neg\mathbf{T}$

≥1: $((p \equiv q) \sim \mathbf{T}) \wedge ((r \equiv s) \sim \mathbf{T}) \supset ((p \geq r) \equiv (q \geq s))$

≥2: $p \geq \bot$

≥3: $(p \geq q) \vee (q \geq p)$

R≥1: $p \vdash (p \sim \mathbf{T})$

R≥2: $p_1,\ldots,p_m \mathbb{E}\, q_1,\ldots,q_m \vdash (p_1 \geq q_1) \wedge\ldots\wedge (p_{m-1} \geq q_{m-1}) \supset (q_m \geq p_m)$

The symbol '$\mathbb{E}$' in R≥2 refers to Segerberg's relation of strict equivalence (cf. [4]), and the antecedent of this rule roughly says that for logical reasons exactly as many sentences from $\{p_1,\ldots,p_m\}$ must be true as sentences from $\{q_1,\ldots,q_m\}$.

Within the language of $\geq$, one may define both a strong and a weak classificatory concept of being probable by

Def. 1 a) $\mathrm{C}p =_{\mathrm{df}} (p \sim \mathbf{T})$

b) $\mathrm{B}p =_{\mathrm{df}} (p > \neg p)$.

The logic of $\geq$ then induces a logic of C which turns out to be isomorphic to Lemmon's alethic modal calculus D (cf. [1], p. 183). Now, interestingly, the logic of the weaker concept B may also be completely axiomatized. The following set of principles (added to PC) will do:

B1: $\mathrm{B}p \supset \neg\mathrm{B}\neg p$

B2: $\mathrm{B}\mathbf{T}$

RB1: $p \supset q \vdash \mathrm{B}p \supset \mathrm{B}q$

RB2: $p_1,\ldots,p_m \mathbb{E}\, q_1,\ldots,q_m \vdash \mathrm{B}p_1 \wedge \neg\mathrm{B}\neg p_2 \wedge \ldots \wedge \neg\mathrm{B}\neg p_m \supset (\mathrm{B}q_1 \vee \ldots \vee \mathrm{B}q_m)$.

To show the adequacy of this system, we first need a *Representation theorem for weak classificatory probability structures* (WCPS):

A WCPS is a structure $\langle S, \Phi\rangle$, where S is a finite, non-empty set and Φ is a set of subsets of S satisfying the following conditions:

a) $S \in \Phi$;

b) If $X \in \Phi$, then $\overline{X} \notin \Phi$;

c) If $X \subseteq Y$ and $X \in \Phi$, then $Y \in \Phi$;

d) For every $m \in \mathbb{N}$ ($m \geq 1$) and for all subsets $X_1,\ldots,X_m$, $Y_1,\ldots,Y_m$ of S: If for every $s \in S$: $\Sigma_{1\leq i\leq m} X_i\hat{}(s) = \Sigma_{1\leq i\leq m} Y_i\hat{}(s)$ (where $X\hat{}$ is the characteristic function of the set X), then $X_1 \in \Phi$ and $\overline{X}_2 \notin \Phi$ and … and $\overline{X}_m \notin \Phi$ entails that $Y_1 \in \Phi$ or … or $Y_m \in \Phi$.

THEOREM

If $\langle S, \Phi\rangle$ *is a WCPS, then there exists (at least one) probability measure P on the powerset of S such that* $P(X) > ½$ iff $X \in \Phi$.

A proof of this representation theorem is given in [3], appendix to section 6.4. Section 6.4 itself contains a derived proof that the above-stated B-logic PC+{B1, B2, RB1, RB2} is sound and complete with respect to the subsequent semantics which is induced by Gärdenfors's definition ([1], p. 176) of a probability model for $\geq$:

Def.2 A probabilistic B-model is a structure $\langle U, P, V\rangle$, where

a) U is a non-empty set (of worlds);

b) P is a function which assigns to every $u \in U$ a probability measure P_u on the powerset of U;

c) V is a two-place valuation function which assigns to every $u \in U$ and every sentence p a truth-value **t** or **f** such that

1) $V(u, \neg p) = \mathbf{t}$ iff $V(u,p) = \mathbf{f}$;

2) $V(u, p \supset q) = \mathbf{f}$ iff $V(u, p) = \mathbf{t}$ but $V(u, q) = \mathbf{f}$;

3) $V(u, \mathrm{B}p) = \mathbf{t}$ iff $P_u([p]) > 1/2$, where $[p] =_{df} \{u' : V(u', p) = \mathbf{t}\}$.

If the probability concepts ≥, C and B are interpreted subjectively, it will be quite natural to consider also iterated probability statements. Gärdenfors's axiom ([1], p. 181):

≥4: $(p \geq q) \equiv ((p \geq q) \sim \mathbf{T}) \wedge \neg(p \geq q) \equiv ((p \geq q) \sim \perp)$

in conjunction with Def.1 entails the following principles for the logic of 'is probable':

B3: $\mathrm{B}p \supset \mathrm{BB}p$

B4: $\neg\mathrm{B}p \supset \mathrm{B}\neg\mathrm{B}p$

B5: $\mathrm{B}(\mathrm{B}p \supset q) \supset (\mathrm{B}p \supset \mathrm{B}q)$.

Since under a subjective reading of ≥, Bp appears to be analytically equivalent to believing that p, {B1-B5, RB1, RB2} may be considered as a *logic of belief*. In [3] it has been shown that this logic is sound and complete with respect to universal B-models, i.e. probabilistic B-models for which $P_u = P_{u'}$, for every $u, u' \in U$.

On the other hand, Cp – subjectively interpreted – says that someone is (completely sure of, or) convinced that p. Since the C-logic induced by {≥1-≥4, R≥1, R≥2} has been shown to be isomorphic to Lemmon's alethic modal calculus DE4 (cf. [2]), we have two interesting systems of doxastic logic: DE4 as the logic of conviction and {B1-B5, RB1, RB2} as a logic of belief. To conclude, I wish to point out that a unified system of doxastic logic may be obtained by adding to PC the following set of principles:

D1: $\mathrm{C}p \supset \mathrm{B}p$

B1: $\mathrm{B}p \supset \neg\mathrm{B}\neg p$

C2: $\mathrm{C}(p \wedge q) \supset \mathrm{C}p \wedge \mathrm{C}q$

B2*: $\mathrm{B}(p \wedge q) \supset \mathrm{B}p \wedge \mathrm{B}q$

D2: $\mathrm{C}(p_1,\ldots,p_m \,\mathbb{E}\, q_1,\ldots,q_m) \supset ((\neg\mathrm{B}\neg p_1 \wedge \neg\mathrm{B}q_1 \vee \mathrm{C}p_1 \vee \mathrm{C}\neg q_1) \wedge \ldots \wedge (\neg\mathrm{B}\neg p_{m-1} \wedge \neg\mathrm{B}q_{m-1} \vee \mathrm{C}p_{m-1} \vee \mathrm{C}\text{-}q_{m-1}) \supset (\mathrm{B}p_m \supset \mathrm{B}q_m) \wedge (\mathrm{C}p_m \supset \mathrm{C}q_m))$

D3: $\mathrm{B}p \supset \mathrm{CB}p$

D4: $\neg\mathrm{B}p \supset \mathrm{C}\neg\mathrm{B}p$

C3: $\mathrm{C}p \supset \mathrm{CC}p$

C4: $\neg\mathrm{C}p \supset \mathrm{C}\neg\mathrm{C}p$

RC: $p \vdash \mathrm{C}p$

Again, a completeness proof may be found in [3]. Furthermore, this system could even been shown to be decidable.

References

[1] Gärdenfors, P., ‘Qualitative Probability as an Intensional Logic’. *Journal of Philosophical Logic* 4, (1975), 171-185.

[2] Lenzen, W., ‘On DE4 – The Logic of Conviction’. To appear in *Theoria*.

[3] Lenzen, W., *Glauben, Wissen und Wahrscheinlichkeit*. Vienna (Springer Verlag), 1980.

[4] Segerberg, K., ‘Qualitative Probability in a Modal Setting’. *Proceedings of the 2nd Scandinavian Logic Symposium* (ed. Fenstad), Amsterdam, 1971.