In [2], modal systems S4.1.4 and S4.021 have been introduced as the result of restricting the proper axioms of S4.4 and S4.04, i.e.,

\[ R1 \quad p \supset (MLp \supset Lp) \]
\[ L1 \quad p \supset (LMLp \supset Lp), \]

R1.3 \quad (p \supset Lp) \supset (ML(p \supset Lp) \supset L(p \supset Lp))

L1.3 \quad (p \supset Lp) \supset (LML(p \supset Lp) \supset L(p \supset Lp)),

respectively. Since R1.3 could be proven to be logically weaker than R1, the author thought it “very probable that L1.3 [would similarly] not entail L1” ([2], p. 162). Also, since S4.021 could be proven to contain the strongest proper subsystem of S4.04, viz. S4.02, properly, the author thought (though rather diffidently) that S4.1.4 might likewise contain the strongest proper subsystem of S4.4, viz. S4.1.2, properly. The aim of this note is to disprove these two assumptions.

As chance would have it, the former assumption which seemed to be the more likely one turned out to be somewhat easier to refute than the latter. The following rather straightforward derivation shows that, even in the field of S2, L1.3 entails L1:

\[
\begin{align*}
(1) \quad (\neg p \supset L\neg p) & \supset (LML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p)) & \text{L1.3, p/\neg p} \\
(2) \quad p \supset (\neg p \supset L\neg p) & & \text{PC} \\
(3) \quad LMLp \supset LML(\neg p \supset L\neg p) & & \text{S2°} \\
(4) \quad p \supset (LMLp \supset L(\neg p \supset L\neg p)) & & (1)-(3) \\
(5) \quad L(\neg p \supset L\neg p) \supset L(Mp \supset p) & & \text{S1°} \\
(6) \quad L(Mp \supset p) \supset (LMp \supset Lp) & & \text{S2°} \\
(7) \quad LMLp \supset LMp & & \text{S2} \\
L1 \quad p \supset (LMLp \supset Lp) & & (4)-(7)
\end{align*}
\]

Hence S4.021 = \{S4; L1.3\} \supset \{S4; L1\} = S4.04.

With respect to R1.3, we obtain in an analogous way:

\[
\begin{align*}
(8) \quad (\neg p \supset L\neg p) & \supset (ML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p)) & \text{R1.3, p/\neg p} \\
(9) \quad MLp \supset ML(\neg p \supset L\neg p) & & \text{S2°} \\
(10) \quad p \supset (MLp \supset L(\neg p \supset L\neg p)) & & (8), (2), (9) \\
(11) \quad p \supset (MLp \supset (LMp \supset Lp)) & & (10), (5), (6)
\end{align*}
\]

But from (11) we cannot further infer R1, because, unlike LMLp, MLp does not (in the field of S4) entail LMp. Formula (11) does, however, entail R1.3 itself! It is only necessary to note that

\[
(12) \quad LM(p \supset Lp)
\]

is a theorem of S4 and that

\[
(13) \quad (p \supset Lp) \supset (ML(p \supset Lp) \supset (LM(p \supset Lp) \supset L(p \supset Lp)))
\]
follows from (11) by substituting \( \frac{p}{p} \supset Lp \). The conjunction of (12) and (13) trivially entails \( \textbf{R1.3} \), which is thus seen to be inferentially equivalent, in the field of S4, to (11).

Now, the proper axiom of S4.01,
\[
\textbf{Γ1} \quad MLP \supset (LMp \supset LMLp),
\]
has been shown by Goldblatt (cf. [1], p. 568) to follow from the proper axiom of S4.1,
\[
\textbf{N1} \quad L(L(p \supset Lp) \supset p) \supset (MLp \supset p);
\]
hence \( \textbf{Γ1} \) is a fortiori provable in S4.1.2 = S4.1 + \( \textbf{L1} \). But (11) follows immediately from \( \textbf{L1} \) in conjunction with \( \textbf{Γ1} \); thus both (11) and \( \textbf{R1.3} \) are theorems of S4.1.2. Since, conversely, \( \textbf{R1.3} \) has been proven to entail both \( \textbf{N1} \) and \( \textbf{L1} \) (cf. [2], p. 161), it follows that S4.1.2 = {S4; \( \textbf{N1} \); \( \textbf{L1} \}) \supset \{S4; (11)\} \supset \{S4; \textbf{R1.3}\} = S4.1.4.\(^1\)

REFERENCES


\(^1\) Another proof may be found in section 1 of [3] which presents a considerable generalization of the investigations made in [2]. Similar proofs that S4 + \( \textbf{R1.3} \) = S4.1.2 and that S4 + \( \textbf{L1.3} \) = S4.04 have been reported to the author by Mr. Steven Schmidt in a letter of Feb. 5, 1978.