S4.1.4 = S4.1.2 and S4.021 = S4.04

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In [2], modal systems S4.1.4 and S4.021 have been introduced as the result of restricting the proper axioms of S4.4 and S4.04, i.e.,

R1 $p \supset (MLp \supset Lp)$
L1 $p \supset (LMLp \supset Lp)$,

to

R1.3 $(p \supset Lp) \supset (ML(p \supset Lp) \supset L(p \supset Lp))$
L1.3 $(p \supset Lp) \supset (LML(p \supset Lp) \supset L(p \supset Lp))$,

respectively. Since **R1.3** could be proven to be logically weaker than **R1,** the author thought it “very probable that **L1.3** [would similarly] not entail **L1**” ([2], p. 162). Also, since S4.021 could be proven to contain the strongest proper subsystem of S4.04, viz. S4.02, *properly*, the author thought (though rather diffidently) that S4.1.4 might likewise contain the strongest proper subsystem of S4.4, viz. S4.1.2, properly. The aim of this note is to disprove these two assumptions.

As chance would have it, the former assumption which seemed to be the more likely one turned out to be somewhat easier to refute than the latter. The following rather straight-forward derivation shows that, even in the field of S2, **L1.3** entails **L1**:

(1)	$(\neg p \supset L\neg p) \supset (LML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p))$	**L1.3**, $p/\neg p$
(2)	$p \supset (\neg p \supset L\neg p)$	**PC**
(3)	$LMLp \supset LML(\neg p \supset L\neg p)$	S2°
(4)	$p \supset (LMLp \supset L(\neg p \supset L\neg p))$	(1)-(3)
(5)	$L(\neg p \supset L\neg p) \supset L(Mp \supset p)$	S1°
(6)	$L(Mp \supset p) \supset (LMp \supset Lp)$	S2°
(7)	$LMLp \supset LMp$	S2
L1	$p \supset (LMLp \supset Lp)$	(4)-(7)

Hence S4.021 = $\{$S4; **L1.3**$\}$ $\leftrightarrows$ $\{$S4; **L1**$\}$ = S4.04.

With respect to **R1.3**, we obtain in an analogous way:

(8)	$(\neg p \supset L\neg p) \supset (ML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p))$	**R1.3**, $p/\neg p$
(9)	$MLp \supset ML(\neg p \supset L\neg p)$	S2°
(10)	$p \supset (MLp \supset L(\neg p \supset L\neg p))$	(8), (2), (9)
(11)	$p \supset (MLp \supset (LMp \supset Lp))$	(10), (5), (6)

But from (11) we cannot further infer **R1**, because, unlike $LMLp$, MLp does not (in the field of S4) entail LMp. Formula (11) does, however, entail **R1.3** itself! It is only necessary to note that

(12) $LM(p \supset Lp)$

is a theorem of S4 and that

(13) $(p \supset Lp) \supset (ML(p \supset Lp) \supset (LM(p \supset Lp) \supset L(p \supset Lp)))$

follows from (11) by substituting $p/p \supset Lp$. The conjunction of (12) and (13) trivially entails **R1.3**, which is thus seen to be inferentially equivalent, in the field of S4, to (11).

Now, the proper axiom of S4.01,

Γ1 $MLp \supset (LMp \supset LMLp)$,

has been shown by Goldblatt (*cf.* [1], p. 568) to follow from the proper axiom of S4.1,

N1 $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$;

hence **Γ1** is *a fortiori* provable in S4.1.2 = S4.1 + **L1**. But (11) follows immediately from **L1** in conjunction with **Γ1**; thus both (11) and **R1.3** are theorems of S4.1.2. Since, conversely, **R1.3** has been proven to entail both **N1** and **L1** (*cf.* [2], p. 161), it follows that S4.1.2 = {S4; **N1**; **L1**} ⇆ {S4; (11)} ⇆ {S4; **R1.3**} = S4.1.4.[1]

REFERENCES

[1] Goldblatt, R. I., "A new extension of S4," *Notre Dame Journal of Formal Logic,* vol. XIV (1973), pp. 567-574.

[2] Lenzen, W., "On some substitution instances of **R1** and **L1**," *Notre Dame Journal of Formal Logic,* vol. XIX (1978), pp. 159-164.

[3] Lenzen, W., "Beschränkte und unbeschränkte Reduktion von Konjunktionen von Modalitäten in S4," in *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 26 (1980), pp. 131-143.

[1] Another proof may be found in section **1** of [3] which presents a considerable generalization of the investigations made in [2]. Similar proofs that S4 + **R1.3** = S4.1.2 and that S4 + **L1.3** = S4.04 have been reported to the author by Mr. Steven Schmidt in a letter of Feb. 5, 1978.