ON SOME SUBSTITUTION INSTANCES OF **R1** AND **L1**

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A study of the epistemic correlates of the modal systems between S4 and S5, [4], has drawn my interest to certain modifications of the “factoring” axioms (*cf.* [11])[1]

R1 $p \supset (MLp \supset Lp)$
L1 $p \supset (LMLp \supset Lp)$

which characterize S4.4 and S4.04, respectively. The following substitution instances turned out to be particularly interesting:

R1.1 $Mp \supset (MLMp \supset LMp)$
R1.2 $(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$
R1.3 $(p \supset Lp) \supset (ML(p \supset Lp) \supset L(p \supset Lp))$
L1.1 $Mp \supset (LMLMp \supset LMp)$
L1.2 $(Lp \supset Lq) \supset (LML(Lp \supset Lq) \supset L(Lp \supset Lq))$
L1.3 $(p \supset Lp) \supset (LML(p \supset Lp) \supset L(p \supset Lp))$

In this note I want to investigate the results of adding these formulae as new axioms to the base of S4 (with a primitive rule of Necessitation). It will be shown that

(i) S4 + **R1.1** is deductively equivalent to S4.2;
(ii) S4 + **R1.2** is deductively equivalent to S4.3.2;
(iii) S4 + **R1.3** is a new system properly between S4.4 and S4.1.2, or else **R1.3** is a new proper axiom of S4.1.2;
(iv) S4 + **L1.2** is a new system properly between S4.04 and S4 and properly between S4.3.2 and S4;
(v) S4 + **L1.3** is a new system properly between S4.04 and S4.02, or else **L1.3** is a new proper axiom of S4.04.

(***A***) It is well known (*cf.* [1], p. 252) that in the field of S4 the proper axiom of S4.2,

G1 $MLp \supset LMp$,

entails and is entailed by

G2 $MLp \supset LMLp$.

Substitution $p/\neg p$ in **G2** yields

(1) $ML\neg p \supset LML\neg p$

from which

(2) $\neg LML\neg p \supset \neg ML\neg p$,

i.e.

(3) $MLMp \supset LMp$

and thus

R1.1 $Mp \supset (MLMp \supset LMp)$

follows truth-functionally. Hence S4 + **R1.1** is contained in S4.2. Conversely, **G1** is easily seen to follow from **R1.1** in conjunction with the following two S2-theorems:

[1] I assume the reader is familiar with the literature cited in this note, especially with [5] and [6].

(4) $MLp \supset Mp$
(5) $MLp \supset MLMp$.

Hence (i), i.e., **R1.1** is another new proper axiom of S4.2.

(***B***) That, in the field of S4, **R1.2** entails the proper axiom of S4.3.2,

F1 $L(Lp \supset q) \vee (MLq \supset p)$,

can be seen as follows:

(6)	$\neg p \supset (Lp \supset Lq)$	S1
(7)	$MLq \supset ML(Lp \supset Lq)$	S4°
(8)	$\neg(MLq \supset p) \supset ((Lp \supset Lq) \wedge ML(Lp \supset Lq))$	(6), (7)
(9)	$(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$	**R1.2**
(10)	$\neg(MLq \supset p) \supset L(Lp \supset Lq)$	(8), (9)
(11)	$L(Lp \supset Lq) \supset L(Lp \supset q)$	S1
F1	$L(Lp \supset q) \vee (MLq \supset p)$	(10), (11)

Hence S4 + **R1.2** contains S4.3.2. For the converse, note that **F1** is known to be inferentially equivalent to

F2 $L(Lp \supset Lq) \vee L(LMLq \supset Lp)$

(*cf*. [10], p. 296), and that in S4.3.2 (which contains S4.2) **G2** is derivable. Moreover, as Zeman has pointed out in [11], in S4.2 (and hence in S4.3.2) *ML* distributes over implications. Thus in particular we have

(12)	$ML(Lp \supset Lq) \supset (MLLp \supset MLLq)$.	S4.2

Now:

(13)	$(MLLp \supset MLLq) \supset (\neg MLLq \supset \neg MLLp)$	**PC**
(14)	$\neg MLq \supset \neg MLLq$	S2
(15)	$\neg LMLq \supset \neg MLq$	**G2**
(16)	$\neg MLLp \supset \neg MLp$	S4°
(17)	$\neg MLp \supset L(Lp \supset Lq)$	S2°
(18)	$ML(Lp \supset Lq) \supset (\neg LMLq \supset L(Lp \supset Lq))$	(12)-(17)

Furthermore we have:

(19)	$Lp \supset ((Lp \supset Lq) \supset L(Lp \supset Lq))$	S4°
(20)	$\neg Lp \supset (LMLq \supset L(Lp \supset Lq))$.	**F2**

(18), (19) + (20) truth-functionally entail

R1.2 $(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$.

Hence (ii), i.e., **R1.2** is another new proper axiom of S4.3.2.

(***C***) The subsequent deduction shows that

S4.1.4 = S4 + **R1.3**

is an extension of Zeman's S4.04:

(21)	$p \supset (\neg p \supset L\neg p)$	**PC**
(22)	$MLp \supset ML(\neg p \supset L\neg p)$	S4°
(23)	$LMLp \supset MLp$	S1
(24)	$p \supset (LMLp \supset (\neg p \supset L\neg p) \wedge ML(\neg p \supset L\neg p))$	(21)-(23)

(25)	$(\neg p \supset L\neg p) \supset (ML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p))$	**R1.3**
(26)	$L(\neg p \supset L\neg p) \supset (LMp \supset Lp)$	S2°
(27)	$LMLp \supset LMp$	S2
L1	$p \supset (LMLp \supset Lp)$	(24)-(27)

Moreover, S4.1.4 also is an extension of S4.1 = S4 +

N1 $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$,

as is proven by the following deduction:

(28)	$MLp \supset ML(p \supset Lp)$	S4°
(29)	$\neg p \supset (p \supset Lp)$	**PC**
(30)	$\neg(MLp \supset p) \supset ((p \supset Lp) \wedge ML(p \supset Lp))$	(28), (29)
(31)	$(p \supset Lp) \wedge ML(p \supset Lp) \supset L(p \supset Lp)$	**R1.3**
(32)	$\neg(MLp \supset p) \supset \neg(L(p \supset Lp) \supset p)$	(30), (31)
(33)	$\neg(L(p \supset Lp) \supset p) \supset \neg L(L(p \supset Lp) \supset p)$	S1
N1	$L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$	(32), (33)

Hence we may conclude that S4.1.4 is also an extension of S4.1.2 = S4.1 + **L1** (*cf*. [7], p. 383). It is easily checked that matrix $\mathfrak{M}_5$ (in [6], p. 350) verifies **R1.3**. Since $\mathfrak{M}_5$ is known to reject S4.2 (*cf*. [7] and [6]), S4.1.4 must be properly included in S4.4. Hence (iii).

(***D***) Since

(34) $LMLMp \supset LMp$

is a well-known S4-theorem (*cf*. [3], p. 47), **L1.1** is of no further interest.

(***E***) However,

S4.03 = S4 + **L1.2**

is an interesting new system. Until presently, the only system known to be contained both in S4.3.2 and in S4.04 was S4 itself. But S4.03 ≠ S4! Sobociński's matrix $\mathfrak{M}_4$ ([6], p. 350) falsifies **L1.2** for, e.g., $p/5$, $q/2$: $(L5 \supset L2) \supset (LML(L5 \supset L2) \supset L(L5 \supset L2)) = (5 \supset 6) \supset LML(5 \supset 6) \supset L(5 \supset 6)) = 2 \supset (LML2 \supset L2) = 2 \supset (LM6 \supset 6) = 2 \supset (L1 \supset 6) = 2 \supset (1 \supset 6) = 2 \supset 6 = 5$. Since $\mathfrak{M}_4$ validates both **N1** and the proper axiom of S4.3,

D2 $L(Lp \supset Lq) \vee L(Lq \supset Lp)$,

(*cf*. [10], p. 297, [5], p. 310), S4.03 properly contains S4 but is not contained in S4.3.1. We know from [9], p. 382, that S4.02 = S4 +

Ł1 $L(L(p \supset Lp) \supset p) \supset (LMLp \supset p)$

is not contained in S4.3.2; since, furthermore, S4.01 = S4 +

Γ1 $MLp \supset (LMp \supset LMLp)$

is not contained in S4.04 (*cf*. [2], p. 569), it follows that S4.03 does not contain any extension of S4 known so far (including the new system S4.021 to be defined in (***F***) below). Hence (iv).

(***F***) Consider now

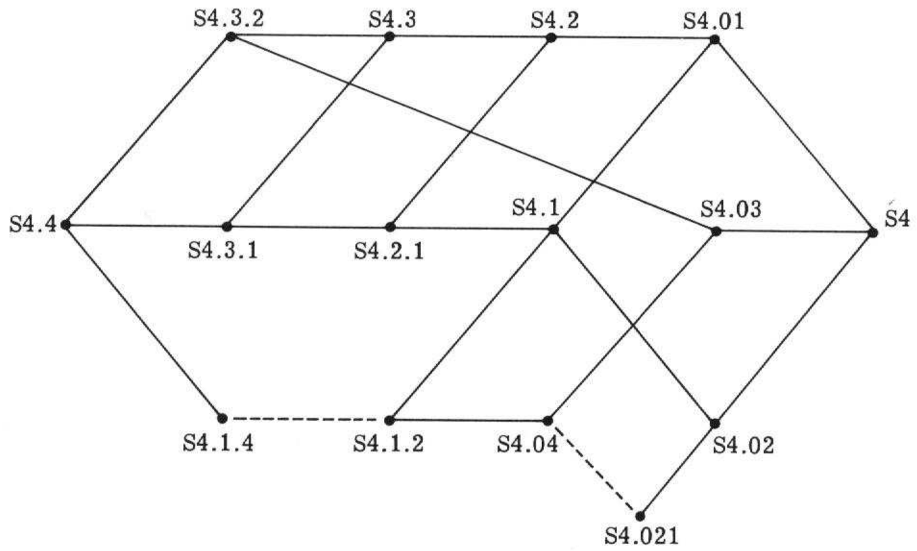
S4.021 = S4 + **L1.3**!

Since we have

(35) $LMLp \supset LML(p \supset Lp)$

as a theorem of S4°, and since **Ł1** "relates" to **N1** as **L1.3** "relates" to **R1.3**, the proof given in (***C***) showing that **R1.3** entails **N1** immediately transforms itself into a proof showing that analogously **L1.3** entails **Ł1.** Hence S4.021 is an extension of S4.02. It is a proper one, because matrix $\mathfrak{M}_9$ ([6], p. 350) verifies **Ł1** (*cf.* [9], p. 381) but falsifies **L1.3** for *p*/13: $(13 \supset L13) \supset (LML(13 \supset L13) \supset L(13 \supset L13)) = (13 \supset 16) \supset (LML(13 \supset 16) \supset L(13 \supset 16)) = 4 \supset (LML4 \supset L4) = 4 \supset (LM12 \supset 12) = 4 \supset (L1 \supset 12) = 4 \supset (1 \supset 12) = 4 \supset 12 = 9$. Since **R1.3** does not entail **R1**, it is very probable that **L1.3** does not entail **L1** either, but I have no proof for this assumption. However, (v) is without doubt the case.

(***G***) The following updated diagram:



visualizes the relations among the systems between S4.4 and S4[2]; the broken line indicates that the respective containment has not yet been proven to be proper.

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[2] The remaining system between S4 and S5, forming the so-called *Z*-family, (*cf.* [8]), are neglected here.

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