1 Introduction

In recent years, various logics have been developed which deviate from the classical propositional calculus (PC) typically with regard to negation. Thus, not only the well-known calculi of intuitionistic logic, but also less-known systems of paraconsistent logic basically adopt the „uncritical“ operators of conjunction, ∧, and disjunction, ∨, while rejecting the seemingly all too simple, two-valued operator of negation, ¬. In what follows ‘¬’ will be used as a symbol for arbitrary other negation-operators. Furthermore, given a certain logic L, we use ‘ ├ L p’ to denote the syntactic relations of provability and deducibility (or derivability) in L as they are determined by the axioms and rules of deduction of L. Thus ‘ ├ L p’ means that p is provable in L, while ‘p₁, ..., pₙ ├ L q’ expresses that, in L, the conclusion q can be derived from the premises p₁, ..., pₙ. The corresponding semantic relations ‘{p₁, ..., pₙ} logically imply q’ and ‘p is logically true’ (in L) are formalised by ‘p₁, ..., pₙ ├ L q’, and ‘ ├ L p’, respectively. The subscript ‘L’ will often be dropped for convenience when it is clear which logic L we are talking about.

In what follows we presuppose an old-fashioned or „classical“ understanding of logical implication in the sense of the subsequent condition:

\[(IMP) \quad \text{Whatever the details of the semantics of L may be, } p₁, ..., pₙ \models L q \text{ holds if and only if the consequent q necessarily becomes true (or gets assigned some other „distinguished“ true-like value) once all of the antecedent formulae p₁, ..., pₙ are true (or have been assigned some other „distinguished“ truth-value).} \]

Thus we do not restrict our considerations only to logics which can be characterised by a classical two-valued semantics. But if one employs a three-, or four-, or ... n-valued or any other kind of semantics, we still insist on the core-idea of logical implication by requiring

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* As the ‘II’ is meant to indicate, the present paper continues my former study of „Necessary Conditions for Negation Operators“*. Actually sections 1-4 below present a slightly revised version of the material first put forward in [Lenzen 1996]. Section 5, however, is entirely new and may be regarded as the completion of my research on non-classical negation which had started with an investigation of intuitionistic negation in [Lenzen 1991]. As I mentioned in the opening fn. of [Lenzen 1996] the topic of paraconsistent negation at that time still needed „to be elaborated and improved before appearing somewhen elsewhere“*. I am grateful to Dov Gabbay for inviting me to write this article for the Handbook thus helping to make this announcement come true.
(i) that one retains at least a counterpart — „true“ — of the classical truth-value ‘true’, and
(ii) that a proposition $q$ logically follows from a set of propositions \{${p_1, \ldots, p_n}$\} just in case that
$q$ cannot fail to have this distinguished value „true“ if all of the $p_i$ are „true“.

This requirement entails that, for each logic $L$ to be considered here, the relation of logical
implication must in particular be reflexive, $p \models_L p$, and monotonic, i.e. whenever a
conclusion $q$ logically follows from a certain set of premises \{${p_1, \ldots, p_n}$\}, the same conclusion
follows also from any larger set \{${p_1, \ldots, p_n, p_{n+1}, \ldots, p_m}$\}. Clearly, if $q$ is „true“ whenever all
sentences from \{${p_1, \ldots, p_n}$\} are „true“, then a fortiori $q$ must be „true“ if all sentences from
\{${p_1, \ldots, p_n, p_{n+1}, \ldots, p_m}$\} are „true“. Since in a complete and sound logic $L$, a consequent formula
$q$ can be syntactically derived from a set of antecedent formulas \{${p_1, \ldots, p_n}$\} if and only if the
conclusion $q$ logically follows from $p_1, \ldots, p_n$, the deducibility-relation $\vdash_L$ of every logic also
has to satisfy the laws of reflexivity and monotonicity:

(REFL) \hspace{1cm} p \vdash_L p

(MONO) \hspace{1cm} If $p_1, \ldots, p_n \vdash_L q$, then $p_1, \ldots, p_n, p_{n+1}, \ldots, p_m \vdash_L q$.

What does this restriction, however, mean for so-called „non-monotonic logics“ as they have
been developed mainly in the field of Artificial Intelligence (cf. [Schaub 1997])? Now, a
proponent of MONO need not deny that there are meaningful forms of non-monotonic
inferences or non-monotonic reasoning. For instance, the basic idea of „default-reasoning“
may be described as follows. From the premises $p_1, \ldots, p_n$ one may conclude by way of
default-reasoning that $q$, if, in the absence of further contrary evidence, $q$ can „normally“ be
expected to be true if the premises $p_1, \ldots, p_n$ all are known to be true. But when certain
„unnormal“ counter-evidence $p_{n+1}, \ldots, p_m$ turns up, it may no longer be reasonable to expect
that $q$; instead it may then be „normal“ to expect that $\neg q$.

Hence the „logic“ of default-reasoning, $D$, is non-monotonic in the strong sense that $p_1, \ldots, p_n \models_D q$ is compatible with
$p_1, \ldots, p_n, p_{n+1}, \ldots, p_m \models_D \neg q$. An advocate of the old-fashioned concept of logical implication
therefore need not protest against the very construction of self-consistent calculi $L$, in which
the inference relation $\models_L$ is non-monotonic; and he may perhaps also not protest if such non-
monotonic calculi nevertheless are called „logics“. After all there is no copyright for the word
‘logic’, and during the long history of logic many strange theories have come to be called by
that name. A proponent of the semantic principle IMP and of its syntactic corollary MONO

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1 A somewhat worn-off example is the default inference from the only premise $p_1 = \text{‘Tweety is a bird’}$ to the conclusion $q = \text{‘Tweety can fly’}$, which no longer remains valid once the additional premise is added $p_2 = \text{‘Tweety is a penguin’}$.
must, however, insist on one decisive point, namely that the inference relation of a non-
monotonic „logic“ never constitutes a relation of logical implication! This may be trivial, and
in the past logicians usually appear to have been well aware of this fact. For instance, when
philosophers like Hans Reichenbach or Rudolf Carnap constructed their systems of „inductive
logic“, they would never have been mislead to think that inductive inferences might be logical
inferences in the sense of deductive inferences, after all. Hopefully contemporary advocates of
default „logic“ and other systems of non-monotonic reasoning will keep in mind that their „deductions“ are not properly logical consequences either, because these inferences don’t have the character of necessarily making the conclusion „true“ once all the premises are assumed to be „true“. Anyway, in what follows, we consider only logical systems \( L \) the
deducibility relation of which satisfies besides REFL and MONO also the so-called „cut-
rule“:

\[(\text{CUT}) \quad \text{If } \Delta, p \vdash_L q, \text{ and } \Gamma \vdash_L p, \text{ then } \Delta, \Gamma \vdash_L q\]

plus the trivial structural rules of permutation and contraction:

\[(\text{PER}) \quad \text{If } \Delta, p_1, p_2 \vdash_L q, \text{ then } \Delta, p_2, p_1 \vdash_L q\]

\[(\text{CON}) \quad \text{If } \Delta, p, p \vdash_L q, \text{ then } \Delta \vdash_L q.\]

Now, in order to determine whether a certain monadic operator \( \neg \) of such a logic \( L \) really is a
negation, one will primarily consider characteristic axioms or rules of deduction which can be
formulated without the help of additional operators. Some well-known examples are the
principles of double negation introduction and elimination:

\[(\text{DNI}) \quad p \vdash \neg \neg p\]

\[(\text{DNE}) \quad \neg \neg p \vdash p\]

and the so-called ex falso quodlibet which is, however, more appropriately referred to as ‘ex
contradictorio quodlibet’:

\[(\text{ECQ}) \quad p, \neg p \vdash q.\]

If the logic \( L \) contains only few such characteristic principles, it may become difficult, if not
impossible, to decide whether \( \neg \) really is a negation. For instance, if \( \neg \) were characterised in \( L \)
just by the two double negation principles, one couldn’t tell whether \( \neg \) represents a negation or
instead an affirmation, because DNI and DNE hold true both under the interpretation \( \neg p = \neg p \)

\[\Delta \text{ and } \Gamma \text{ are arbitrary finite or infinite sets of formulas. As usual, if } \Delta \text{ is infinite, } \Delta \vdash q \text{ is taken to hold iff there exists a finite subset } \Delta^* \text{ of } \Delta \text{ such that } \Delta^* \vdash q.\]
and under the opposite interpretation \(\neg p = p\). Therefore it may be necessary to investigate also logical implications that hold between „negated“ formulas and certain other formulas containing further operators besides \(\neg\). For convenience we presuppose that each logic \(L\) contains at least the following elementary part of the usual theory of conjunction and disjunction:

\[
\begin{align*}
(p \land q) & \vdash p \\
(p \land q) & \vdash q \\
\text{If } p & \vdash q \text{ and } p \vdash r, \text{ then } p \vdash (q \land r)
\end{align*}
\]

\[
\begin{align*}
p & \vdash (p \lor q) \\
q & \vdash (p \lor q) \\
\text{If } p & \vdash r \text{ and } q \vdash r, \text{ then } (p \lor q) \vdash r.
\end{align*}
\]

The availability of these laws would not be absolutely essential for the main aim of our paper, i.e. for the development of certain conditions of adequacy that any proposed monadic operator has to satisfy in order to be rightly called a negation. It will turn out below that the decisive conditions can in general be formulated without the help of additional operators. However, given the theory of \(\land\) and \(\lor\) in the background of the respective logic \(L\), it is much easier to illustrate the import of these conditions by means of simple examples.

### 2 Conditions of Adequacy

In this section three types of negation principles will be discussed: Unacceptable principles which a logic \(L\) must never satisfy, if its „negation“-operator \(\neg\) is to rate as a real negation; dispensable principles which, though they are valid principles of classical negation, need not necessarily be satisfied by arbitrary other negations; and indispensable principles which a logic \(L\) always has to satisfy if its monadic operator \(\neg\) is to count as a genuine negation. Let us begin by considering some unacceptable principles. If \(L\) contains the elementary theory of conjunction, we would certainly be reluctant to accept a monadic operator \(\neg\) as a real negation if \(L\) were to contain an axiom or rule of deduction according to which the conjunction of two propositions entails the „falsity“ of, say, its left conjunct:

\[
(p \land q) \vdash \neg p.
\]

\[3\text{ In addition } L\text{ may but need not contain a theory of implication which can be either material implication, } \supset, \text{ or strict implication, } \rightarrow. \text{ In this case we assume the implication operator to represent the metalinguistic relation of deducibility in the sense that } \models_L (p \supset q) \text{ or } \models_L (p \rightarrow q) \text{ iff } p \models_L q.\]
Instead it seems reasonable to require that, in any plausible sense of the word ‘false’, the falsity of one conjunct entails the falsity of the entire conjunction. Similarly, if the logic \( L \) contains the classical theory of disjunction, then the monadic operator \( \sim \) would not rate as a real negation, if \( L \) were to contain an axiom or rule of deduction according to which the disjunction of two propositions entails the „falsity“ of one of the disjuncts, say:

\[(UN\ 2)\quad (p \lor q) \models \sim p.\]

If \( L \) contains at least some most elementary theorems following from AND and OR, we could otherwise derive either from UN 1 or from UN 2 the clearly unacceptable principle

\[(UN\ 3)\quad p \models \sim p.\]

Although a self-consistent logic \( L \) may in general contain some propositions \( p \) which logically entail \( \sim p \), it would be very strange indeed if every proposition could be proven to entail its own „falsity“. This would evidently mean that the logic \( L \) itself either is inconsistent, or that the negative formulae \( \sim p \) does not really express the negation of \( p \). Since UN 3 is unacceptable, so is — a fortiori — the following principle according to which every proposition \( p \) would be provably „false“ (in the sense of the non-classical negation operator \( \sim \)):

\[(UN\ 4)\quad \models \sim p.\]

To be sure, if a certain logic \( L^* \) is inconsistent, i.e. if for every proposition \( q \) one has \( \models_{L^*} q \), then the result UN 4 trivially holds, no matter how the negation operator \( \sim \) is interpreted. But it seems uncontroversial to postulate that a self-consistent logic must never satisfy any of the principles UN1, UN2, UN 3, or UN 4. If it is observed that UN 1 - UN 3 are deductively equivalent to each other (provided that \( L \) contains some most elementary laws of conjunction and disjunction)

\[\text{ADQ 1}\quad \text{If } L \text{ is a self-consistent logic, and if } \sim \text{ is a real negation operator, then } L \text{ must not satisfy UN 3, i.e. not every proposition } p \text{ may entail its own negation.}\]

In other words: If a self-consistent logic \( L \) contains a unary operator \( \sim \) such that, for every proposition \( p \), \( \sim p \) logically follows from \( p \), then \( \sim \) does not represent a real negation of \( L \).

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\[\text{Suppose UN 1 to hold in } L, \text{ then we have in particular } (p \land p) \models \sim p \text{ from which one obtains UN 3 via the conjunction law } p \models (p \land p); \text{ conversely, if UN 3 holds in } L, \text{ then one obtains UN 1 by means of the further conjunction law } (p \land q) \models p. \text{ The reader may verify in a similar way that UN 2 is equivalent to UN 3, too.}\]
Next let us consider some characteristic laws of classical negation which seem to be *dispensable* for arbitrary negations. In the literature several non-classical negations ~ have been suggested for which *tertium non datur*

(TND) \[ \models (p \lor \neg p) \]

does not necessarily hold. Let us — preliminarily— refer to such operators as strong negations. From the point of view of classical two-valued semantics, the non-validity of TND for strong negation operators ~ might be described as follows. The fact that p is *not true* does not always guarantee that p is *false* in the strong sense of ~p, because the truth of ~p may require something more than the mere (classical) falsity of p. Intuitionistic negation appears to represent an example of such a strong negation. The intuitionistically negated formula ~p roughly says that p is *provably false* or that p can be shown to entail a contradiction. A related kind of strong negation operator might be defined in the framework of ordinary modal logic with the help of classical negation by:

**Def. 1**  
\[ \sim_s p := \neg p. \]

If the necessity-operator satisfies the usual „truth-axiom“ ( \( p \supset p \) ), then strict or necessary falsity in the sense of Def. 1 entails but is not conversely entailed by ordinary (plain) falsity, \( \neg \). In this sense a strongly negated formula is stronger than a classically negated formula. Therefore strong negations in general do not satisfy TND. Similarly, the principle of double negation elimination, DNE, fails to hold for \( \sim_s p \), since \( \sim_s \sim_s p \), i.e. \( \neg \neg p \) or \( \diamond p \), does not generally entail p.

Other non-classical negations ~ have been suggested for which in particular ECQ does not necessarily hold. Let us — again preliminarily — refer to them as weak negations. The negation operators that have been put forward in some systems of paraconsistent logic appear to represent such weak negations. Another particularly simple example of a weak negation might be defined in the framework of classical modal logic by

**Def. 2**  
\[ \sim_w p := \diamond \neg p. \]

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5 Cf. also the related approach of [Došen 1986].

6 If a modal system contains ( \( \diamond p \supset p \) ) as a theorem, then by the usual rule of necessitation also ( \( \diamond p \supset p \) ) becomes provable. By the law of \( \neg \)-distribution one thus obtains ( \( \neg p \supset p \) ). But in S5 \( \diamond p \) follows already from \( \diamond p \) so that in this system the clearly unvalid principle \( \diamond p \supset p \) would become provable.
Such a weakly negated proposition only maintains that p is possibly false; this, of course, is fairly compatible with the assumption that p is actually true. Since, then, the conjunction of p and \( \sim_w p \) will in general be self-consistent, principle ECQ does not hold for weak negation operators. Similarly, it may be shown that the principle of double negation introduction, DNI, fails to hold for the particular operator \( \sim_w p \). In view of Def. 2, a doubly negated proposition \( \sim_w \sim_w p \) says that p is possibly necessary. However, there exist some normal modal calculi where the actual truth of p does not entail \( \Diamond \ p \). In sum, then, the consideration of strong and weak negation operators suggests that none of the classical principles TND, ECQ, DNI or DNE should be postulated as a general condition of adequacy for arbitrary negations.

Let us now turn to the most important category of principles which are indispensable for any real negation operator. As was mentioned earlier in connection with UN 1, it seems reasonable to require that in any plausible sense of ‘false’, the falsity of one conjunct entails the falsity of the entire conjunction: \( \sim p \vdash \sim(p \land q) \). Similarly it seems indispensable to require that the falsity of a disjunction entails the falsity of each of the disjuncts, say: \( \sim(p \lor q) \vdash \sim p \). As a matter of fact, these two inferences are only special instances of a much more general principle which any monadic operator \( \sim \) apparently has to satisfy in order to be justifiably called a negation. Just as a negative number \( -x \) is the smaller the greater the positive number x is itself, so also — one will want to say — a negative or negated proposition \( \sim p \) is „the falser“ „the truer“ the positive proposition p is itself. Somewhat more exactly: If p is „at most as true“ as q, i.e. if p logically entails q, then conversely q is „at most as false“ as p, i.e. the „falsity“ of the latter proposition q logically entails the „falsity“ of the former proposition p:

\[
\text{(CP 1)} \quad \text{If } p \vdash q, \text{ then } \sim q \vdash \sim p.
\]

Before stating some reasons for accepting CP 1 as an absolutely indispensable principle for negation, let me briefly consider a few arguments that have been raised in the literature against various versions of the law of contraposition. [Da Costa/Wolf 1980: 199] criticise the „axiom

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7 Observe that, with \( \neg p \) substituted for p, the principle \((p \supset \Diamond p)\) entails \((\neg p \supset \Diamond \neg p)\) or, by contraposition, \((\neg \Diamond \neg p \supset \neg \neg p)\), i.e. \((\Diamond p \supset p)\), which was shown in the preceding fn. not to be modally valid.

8 Some further dispensable principles will be discussed in section 4.

9 This core of the traditional theory of „conversio per contrapositionem“ has been formulated by one of the forerunners of modern logic, G.W. Leibniz, in [1984: 522] as follows: „Si ex propositione L [...] sequitur propositio M [...] tunc contra ex falsitate propositionis M sequitur falsitas propositionis L.“ For more details of Leibniz’s logic, cf. [Lenzen 1990].
of contraposition: \((p \supset q) \supset (\neg q \supset \neg p)\)\(^{10}\), because when it is „coupled with double negation [it] leads to a collapse“ of their „dialectical logic“ DL into classical PC. First, however, within the framework of paraconsistent logic this collapse should better be taken as an argument to give up the laws of double negation rather than the principle of contraposition. Second, contraposition plus double negation does not automatically lead to classical logic. Da Costa/Wolf evidently failed to notice the different logical import of their (strong) axiom of contraposition,

\((\text{CP 2})\quad \vdash_{\text{DL}} (p \supset q) \supset (\neg q \supset \neg p)\)

on the one hand, and the above (weak) rule CP 1 on the other. Within the axiomatic framework of a calculus containing the operator of material implication (such as their „dialectical“ system DL), this difference is tantamount to the difference between (strong)

\((\text{CP 3})\quad (p \supset q) \vdash (\neg q \supset \neg p)\),

and (weak)

\((\text{CP 4})\quad \text{If } \vdash (p \supset q), \text{ then } \vdash (\neg q \supset \neg p)\).

Independently of the issue of double negation, CP 3 turns out to be definitely stronger than CP 4 because, in conjunction with the ordinary implication principle \(q \vdash (p \supset q)\), CP 3 entails \(q \vdash (\neg q \supset \neg p)\) and hence by way of modus ponens also the following variant of ECQ: \(q, \neg q \vdash \neg p\).

[Da Costa/Wolf 1980: 201] offered a second argument in favour of rejecting contraposition by maintaining that their „characteristic semantics“ of DL „does not directly justify contraposition“. However, this „semantics“ consists only of a certain set of four-valued matrices which in no way reflects the intended „meaning“ of the paraconsistent operators but which had been constructed in a more or less arbitrary way so as to validate just the theorems of DL. Unfortunately, however, the authors failed to notice that their semantics didn’t fully achieve the job it was designed to do! A closer examination reveals that the matrices in question falsify one axiom of their „dialectical logic“, to wit: If one puts \(p=0\) and \(q=3\), then da Costa/Wolf’s axiom „A12“ – or more exactly its implication from left to right – receives the value \(\neg_{\text{DL}}(0 \vee 3) \supset (\neg_{\text{DL}}0 \wedge \neg_{\text{DL}}3) = \neg_{\text{DL}}3 \supset (0 \wedge 3) = 3 \supset 1 = 1\); but in their semantics 1 is not a designated value!

\(^{10}\) In this as well as in the subsequent quotations I have unified the logical symbolism; in particular, da Costa/Wolf's symbol \(\neg\) has been replaced by \(\sim\).
[Pearce 1992: 67] argues contraposition not to be indispensable for arbitrary negations since a particular type of negation – which he refers to as „hard negation“ – „cannot be contrapositive“. This claim is then substantiated by a counterexample of a logic PL which fails to satisfy the specific principle:

\[(\text{CP} \, 5) \quad \text{If} \, \neg p \vdash_{\text{PL}} q, \text{then} \, \neg q \vdash_{\text{PL}} p.\]

As will be shown in section 4, however, this contraposition principle again is definitely stronger than our basic principle CP 1, because the former unlike the latter entails double negation elimination, DNE. (Conversely, CP 1 plus DNE also entails CP 5). Unfortunately Pearce was bound to overlook the difference between CP 1 and the criticised principle CP 5 because he based his entire investigations on the full theory of double negation.

A similar remark applies to the system Cωωω of [da Costa 1974] in which contraposition is rejected because otherwise – together with some elementary laws of implication – the paraconsistently unwanted principle ECQ would become provable. A short inspection of the relevant derivation given, e.g., in [Hunter 1997: 17] reveals, however, that ECQ will be obtained only in the presence of some strong principle of contraposition such as CP 5 (or da Costa’s axiom \((\neg p \supset q) \supset (\neg q \supset p)\)), but not by means of the weaker CP 1. Hence in order to preserve the main goal of paraconsistent logic and prevent ECQ from becoming a theorem, it is not at all necessary to give up the basic principle of contraposition, CP 1; it is equally possible, and, indeed, much more advisable to dispense with the principle(s) of double negation, DNI and/or DNE, in its stead. This issue will be further discussed in section 5 below.

Let me now present some constructive arguments in favour of CP 1. In ordinary discourse there exist many different forms of affirmation and also many different forms of negation. To maintain, to claim, to believe, to assert, etc. all belong to the former category; and to deny, to doubt, to reject, to disbelieve, to disprove, etc. to the latter. What appears to be common to all forms of negation is that they are somehow opposite to (a corresponding way of) affirmation. Sometimes this opposition can be made explicit by transforming the negation into an affirmation with a negative or negated content; sometimes the negation can be analysed as representing the direct (classical) negation or denial of some affirmative expression. This is not the place to investigate the linguistic relations between arbitrary negative and affirmative expressions in greater detail. Rather I want to inquire in a rather abstract and admittedly speculative way into the logical relationships between affirmations and negations, as they are suggested by analogy to negative and positive expressions in arithmetic.
Elementary mathematics teaches us that the product of two positive numbers, \((+x)(+y)\), as well as the product of two negative numbers, \((-x)(-y)\), always yields a positive number, while \((+x)(-y)\) and \((-x)(+y)\) is negative. Similarly, the affirmation of an affirmative expression and the negation of a negative expression appear to yield an affirmation, while the affirmation of a negation as well as the negation of an affirmation normally represents a negation. In particular, it seems safe to maintain that the classical negation of an affirmative expression itself constitutes a negation, while the classical negation of an arbitrary negation yields an affirmation. E.g., if \(\neg p\) is taken to mean that \(p\) is „false“, or that \(p\) is „not true“, or that \(p\) is „impossible“, or that \(p\) is „unlikely to be true“, or what not, then the classically negated expression \(\neg \neg p\) either says that \(p\) is not false, or that \(p\) is true, or that \(p\) is not impossible, or that \(p\) is not unlikely to be true, etc. All these expressions evidently represent various kinds of an affirmation.

Now, for any affirmative operator \(\Phi_A\), the premise that \(p\) logically entails \(q\) generally seems to warrant that \(\Phi_A(p)\) logically entails \(\Phi_A(q)\), too. To be sure, I have no proof of this very strong principle. In order to lend it at least some credibility, however, let it be pointed out that this law holds not only for each so-called affirmative modality in alethic modal logic (and in related systems for other modal operators), but also, e.g., for probabilistic notions saying that \(p\) is likely, that \(p\) is probable, that \(p\) is certain, etc. If it could thus be taken for granted that affirmations are generally closed under logical implications, CP 1 might be „proved“ as follows. Let \(p, q\) be propositions such that \(p\) logically entails \(q\), and let \(q\) be „false“ in the sense of the operator \(\sim\), i.e. let \(\sim q\) be true. Suppose further that — contrary to CP 1 — \(p\) would not be „false“ in the sense of \(\sim\); i.e. suppose that \(\sim p\) does not hold. This can be expressed metalinguistically by means of classical negation as \(\neg \sim p\). Hence we have in sum the assumption that \(p \models q\); that \(\sim q\); and that \(\neg \sim p\). Now if \(\sim\) really is a negation operator — no matter what particular kind of „negation“ one has in mind — then \(\neg \sim\) certainly will not be a negation itself but instead expresses some sort of affirmation. Thus by our assumption that affirmations are closed under logical implication, \(\neg \sim p\) logically entails \(\neg \sim q\); and hence we would arrive at the classical contradiction that, on the one hand, \(\sim q\), and on the other hand, \(\neg \sim q\).

The „proof“ of CP 1 just given, even if its main idea is basically sound, certainly remains problematic insofar as the central assumption of the logical closure of arbitrary affirmations is in need of justification at least to the same degree as the principle that it purports to justify. Therefore let me sketch another argument in favour of CP 1 which does without the latter
assumption. As was argued above in connection with principle IMP, a logical implication \( p \vDash L q \) must be understood to hold only if the conclusion \( q \) cannot fail to be „true“ once the entailing proposition \( p \) itself is assumed to be „true“. Or, to give a somewhat more formal paraphrase: If \( p \vDash L q \), then, necessarily, (if \( p \) is „true“, then \( q \) is „true“). So, by classical contraposition: If \( p \vDash L q \), then, necessarily, (if \( q \) is not „true“, then \( p \) isn’t „true“ either). But to say that some proposition \( p \) is not „true“ (either in the sense of classical, two-valued semantics or in the sense of some other distinguished true-like value) appears to be tantamount to negating \( p \) (in some way or another). Thus one obtains the following version of CP 1: If \( p \vDash q \), then, necessarily (if \( q \) is „false“, then \( p \) must be „false“, too). I hope these reflections have lent some plausibility to the claim that CP 1 really is an indispensable principle for negation and that therefore a second condition of adequacy may be stated as follows:

\[
\text{(ADQ 2) A unary operator } \sim \text{ is a negation of the logic } L \text{ only if it satisfies CP 1: If } p \vDash L q, \text{ then } \sim q \vDash L \sim p.
\]

Let it be repeated here that CP 1 represents only a weak form of contraposition. In section 4 below, several stronger principles of contraposition will be discussed which do not hold for arbitrary negations. For the moment suffice it to point out that because of CP 1 each negation, although not necessarily being a truth-functional (or extensional) operator, is at least a propositional (or intensional) operator in the following sense: If \( p \) and \( q \) are logically equivalent sentences thus expressing the same proposition, then \( \sim p \) and \( \sim q \) have to be logically equivalent, too:

\[
\text{(EQUI) If } p \vDash \vDash q, \text{ then } \sim p \vDash \vDash \sim q.
\]

Now, even in conjunction with ADQ 1, the necessary condition ADQ 2 is not sufficient for negation operators because CP 1 is satisfied, among others, both by the „tautology-operator“, \( \sim_t \), and by the „contradiction-operator“, \( \sim_c \), which might be defined in the framework of classical PC as follows:

\[
\text{Def. 3 } \sim_t p := (p \lor \neg p) \\
\text{Def. 4 } \sim_c p := (p \land \neg p).
\]

These operators certainly cannot be rated as genuine negations. As a matter of fact, the „tautological negation“ according to Def. 3 is not a negation because, for any proposition \( p \),
the formula \( \neg p \) is provable in \( \text{PC} \). This violates our condition ADQ 1 according to which \( \neg \) is a genuine negation of a logic \( L \) only if not every proposition is provably „false“.

In a similar way the „contradictory negation“ according to Def. 4 should not be regarded as a genuine negation because otherwise no proposition, not even the impossible ones, could be proven to be „false“. One might therefore think of postulating another condition of adequacy as follows:

\[
(\text{ADQ 3}) \quad \text{A unary operator } \neg \text{ is a negation of the logic } L \text{ only if there is at least one proposition } p \text{ such that } \models_L \neg p.
\]

However, this condition appears somewhat problematic because the language of \( L \), even if it contains the theory of conjunction and disjunction, may nevertheless lack the means for expressing a particular proposition \( p \) which is „sufficiently“ false in the sense that \( \neg p \) must be provable in \( L \). To be sure, if \( L \) contains the conceptual means for expressing a classical contradiction such as \((p \land \neg p)\), then we are certainly justified in requiring that this proposition, which according to the familiar law of consistency is provably false in classical logic, \( \models_{\text{PC}} \neg(p \land \neg p) \), must also be provably „false“ in any other logic, \( \models_L \neg(p \land \neg p) \), no matter what particular interpretation is given to the negation-operator \( \neg \). However, in \( L \) one can in general formulate only the corresponding conjunction \((p \land \neg p)\) which may perhaps express no real contradiction at all. Therefore it is not clear whether the „non-classical law of consistency“: \( \models_L \neg(p \land \neg p) \), represents an indispensable principle for arbitrary negations. A more detailed discussion of this principle shall be postponed to the end of the next section.

As regards the aforementioned scruples concerning ADQ 3, however, note that when \( p \) is a theorem of a certain logic \( L \), then, whatever the details of the semantics of \( L \) may be, \( p \) is necessarily „true“ and hence \( p \) cannot be „false“ in any reasonable sense of the word. So if \( p \) is a theorem of \( L \), and if the operator \( \neg \) really represents a negation, then \( \neg p \) expresses sort of a contradiction in \( L \) and must therefore itself be classified as necessarily „false“. This observation justifies to postulate the following principle of double negation introduction for theorems:

\[
(\text{DNIT}) \quad \text{If } \models_L p, \text{ then } \models_L \neg\neg p.
\]

This principle is weaker than the usual law of double negation introduction, DNI, which was argued above not to be indispensable for arbitrary negations. Semantically speaking, the inference \( p \models \neg\neg p \) amounts to the condition that, necessarily, if \( p \) is „true“, so will be \( \neg\neg p \). In

\[\text{Cf. also [Wansing 1993: 142, fn. 12] where it is similarly argued that the „tautology operator“ should not be accepted as a negation.}\]
contrast DNIT only requires that, necessarily, when \( p \) is provably (or necessarily) ,,true“, \(-p\) must be necessarily ,,false“, and hence \(~~p\) must be necessarily ,,true“, too.

Now, each logic \( L \) normally does contain some theorems \( p \). This is trivial whenever \( L \) contains an operator of material or strict implication. Then at least some formulas such as \((p \supset p)\) or \((p \rightarrow p)\) will be provable in \( L \). And if \( L \) contains the theory of conjunction, the rules of inference \( \text{AND} \) can be transformed with the help of ‘\( \supset \)’ or ‘\( \rightarrow \)’ into corresponding theorems such as \( \vdash ((p \land q) \supset p) \) or \( \vdash ((p \land q) \rightarrow q) \), etc. So, in view of DNIT, there normally exist in \( L \) some theorems of the form \( ~~p \), and a fortiori there also exists some \( q \) (namely \( q = ~p \)) such that \( \vdash _L ~q \). In other words, ADQ 3 will normally be satisfied after all. Yet we cannot absolutely exclude the possibility that in an unnormal \( \{ \land, \lor, \neg \} \) -logic \( L \) neither the ,,positive“ formula \((p \lor ~p)\) nor the ,,negative“ formula \(~(p \land ~p)\) nor any other formula is strictly provable\(^1\). Therefore instead of ADQ 3 only the subsequent Fourth Condition of Adequacy shall be postulated:

\[
\text{(ADQ 4)} \quad \text{A unary operator } \sim \text{ is a negation of the logic } L \text{ only if for every proposition } p:\ \text{If } \vdash _L p, \text{ then } \vdash _L ~~p.
\]

3 Gabbay’s ,,Necessary and Sufficient“ Condition for Negation

In a paper of [1988], Dov Gabbay investigated the question ,,What is Negation in a System?“., and he suggested the following answer\(^1\):

\[
\text{(GAB)} \quad 
\text{"~ is a form of negation [...] iff for any } \Delta \text{ and any } p \text{ the following holds: } \\
\Delta \vdash ~p \text{ iff for some } q \text{ such that } \vdash ~q \text{ we have } \Delta, p \vdash q."
\]

The set \( \Delta \) is taken by Gabbay to contain the ,,unwanted“ formulas of the respective logic \( L \), but this heuristic interpretation seems to be quite irrelevant for the logical import of the quoted criterion where \( \Delta \) just functions as a variable for arbitrary sets of sentences. In order to be able to compare the general criterion GAB with our own conditions of adequacy, consider

\(^{12}\) The ,,minimale Aussagenlogik“ of [Kutschera 1985: 30] represents such a counterexample: ,,In ihr ist insbesondere kein einziger Satz beweisbar“. Interestingly, the failure of \((p \supset p)\) as a theorem is motivated by the observation: ,,wegen der Definition der Implikation durch die Disjunktion wäre \([p \supset p]\) mit dem Prinzip tertium non datur \([~p \lor p]\) äquivalent“. Now if someone wants to give up TND, it would seem to be much more natural to reject the definition of implication in terms of disjunction rather than to dispense with the reflexivity of the implication operator.

\(^{13}\) Cf. [Gabbay 1988: 100]; in the quotation I have replaced some of Gabbay’s symbols by the logical terminology used throughout this paper. In particular I write simply ‘\(\varnothing \vdash p\)’ instead of Gabbay’s ‘\(\varnothing \vdash p\)’. 

13
first the following necessary conditions which are obtained from GAB by setting \( \Delta = \emptyset \), \( \Delta = \{ \neg p \} \), and \( \Delta = \Delta^* \cup \{ r \} \), respectively:

(GAB 1) \( \models \neg \neg p \) iff there is a \( q \) such that \( \models \neg q \) and \( \models q \);

(GAB 2) \( \models \neg p \) \( \models \neg p \) iff there is a \( q \) such that \( \models \neg q \) and \( \models \neg p \), \( \models p \);

(GAB 3) \( \models \Delta^*, r \models \neg p \) iff there is a \( q \) such that \( \models \neg q \) and \( \Delta^*, r \models q \).

The equivalence GAB 1 may be split up into two implications. The first one: ‘If \( \models \neg p \) then there is a \( q \) such that \( \models \neg q \) and \( \models q \)’ holds trivially — just set \( q = p \) ! The converse implication:

(GAB 4) If there is a \( q \) such that \( \models \neg q \) and \( \models q \), then \( \models \neg p \)

says that if \( p \) logically entails a proposition \( q \) which is provably false, then \( p \) must be provably false, too. This principle of contraposition is slightly weaker than the basic requirement CP 1. It will turn out soon, however, that GAB entails another principle of contraposition which is even stronger than CP 1.

In view of the trivial consequence \( \neg p \models \neg p \), it follows from GAB 2 that there exists a proposition \( q \) such that \( \models \neg q \) and \( \models \neg p \), \( \models p \), or — by the laws of conjunction — \( (p \land \neg p) \models q \).

Because of GAB 4, then, \( (p \land \neg p) \) itself is provably false. As was already pointed out in [Wansing 1993: 143], one thus obtains the crucial „law of consistency“

(CONSIS 1) \( \models \neg (p \land \neg p) \).

With the help of CONSIS 1, however, the following stronger principle of contraposition can be proven:

(GAB 5) If \( \Delta, p \models q \), then \( \Delta, \neg q \models \neg p \).

For if \( \Delta, p \models q \), then by the laws of conjunction \( \Delta, p, \neg q \models (q \land \neg q) \); hence there exists an \( r \), namely \( r = (q \land \neg q) \), such that (by CONSIS 1) \( \models \neg r \) and such that \( \Delta, p, \neg q \models r \). GAB 3 thus entitles us to conclude that \( \Delta, \neg q \models \neg p \).

GAB 5 is stronger than our basic principle CP 1 because it not only contains the latter as a special instance for \( \Delta = \emptyset \) but also entails the following principle which, in view of the parallel to so-called „disjunctive syllogism“

(DISSYL) \( (p \lor q), \neg p \models q \).

is perhaps not inappropriately referred to as „conjunctive syllogism“:

(CONSYL) \( p, \neg (p \land q) \models \neg q \).
Clearly, since \( p, q \vdash (p \land q) \), CONSYL immediately follows from GAB 5 by the substitution \((\Delta/p; p/q; q/p\land q)\). Conversely GAB 5 can be derived from CP 1 in conjunction with CONSYL as follows. Let \( \Delta \) be a finite set of formulas such that \( \Delta, p \vdash q \), and let \( r \) abbreviate the conjunction of all elements of \( \Delta \). Hence the premise \( \Delta, p \vdash q \) can be rendered as \( (p \land r) \vdash q \), so that CP 1 gives us \( \neg q \vdash (r \land \neg p) \). By the laws of conjunction one further obtains that \( (r \land \neg q) \vdash \neg r \land (r \land p) \). But, by CONSYL, \( r \land (r \land p) \vdash \neg p \), so that by CUT we also get \( \neg r \land \neg q \vdash \neg p \), i.e. the desired conclusion \( \Delta, \neg q \vdash \neg p \).

Next observe that the general criterion GAB is already satisfied if the special instances CONSIS 1 and GAB 5 hold. On the one hand, if a set of formulae \( \Delta \) entails \( \neg p \), then by the laws of conjunction \( \Delta, p \vdash (p \land \neg p) \). Hence there exists a \( q \), namely \( q = (p \land \neg p) \), such that \( \Delta, p \vdash q \) and such that (by CONSIS 1) \( \vdash \neg q \). On the other hand, if there exists a \( q \) such that \( \vdash \neg q \) and \( \Delta, p \vdash q \), then — because of GAB 5 — \( \Delta, \neg q \vdash \neg p \). Since \( \neg q \) is assumed to be provable, it follows that \( \Delta \vdash \neg p \).

Summing up the results of the foregoing derivations, Gabbay’s „necessary and sufficient“ condition GAB turns out to be „equivalent“\(^{14}\) to CP 1, CONSIS 1, plus CONSYL. However, this set of principles is not entirely sufficient for arbitrary negations because GAB fails to warrant ADQ 1\(^{15}\). On the other hand, it is somewhat doubtful whether the two principles CONSYL and CONSIS 1 (by which GAB surmounts our own ADQ) really are necessary conditions for negation operators. As regards the former, conjunctive syllogism certainly is not indispensable for arbitrary negations because it will be satisfied neither by strong negations à la Def. 1 nor by weak negations à la Def. 2. To wit, if \( \neg p \) is interpreted as \( \neg p \), then the premises of CONSYL, \( p \) and \( \neg(p \land q) \), say that \( p \) is „true“ and that \( (p \land q) \) is necessarily „false“. In order to infer that \( \neg q \), however, one would have to have the stronger

\(^{14}\) ‘Equivalent’ here and in related places is to be understood as ‘equivalent within a logic \( L \) such that (1) \( L \) contains AND and OR; (2) the deducibility relation of \( L \) satisfies the requirements formulated in section 1; and (3) the \( \neg \)-operator of \( L \) satisfies the conditions of adequacy formulated in section 2’.

\(^{15}\) Consider, e.g., the particular logic \( L^* \) in which every negated formula \( \neg p \) is provable but which nevertheless remains self-consistent because \( \neg \) does not satisfy ECQ and thus not every unnegated formula \( q \) is provable in \( L^* \). As the reader may easily verify, the criterion GAB is then trivially satisfied, but the operator \( \neg \) of this logic certainly should not be regarded as a genuine negation. Note, incidentally, that the remaining conditions of adequacy hold on Gabbay’s approach. ADQ 2, i.e. CP 1, is trivial. ADQ 3 immediately follows from CONSIS 1. Furthermore also ADQ 4, i.e. the principle of double negation introduction for theorems follows from CONSIS 1. Take any \( q \) such that \( \vdash \neg q \); by MONO one has both \( p \vdash q \) and \( \neg p \vdash q \), so that elementary contraposition yields \( \neg q \vdash \neg p \) and \( \neg q \vdash \neg \neg p \). By the rules of conjunction and another application of CP 1 one obtains \( \neg(p \land \neg q) \vdash \neg q \); but according to CONSIS 1 \( \neg(q \land \neg p) \) is provable; hence \( \vdash \neg \neg q \).
premise that, besides \( \neg(p \land q) \), also the proposition \( p \) is necessarily „true“.

Similarly, if \( \neg \) is interpreted in the weak sense of Def. 2, then \( p, \neg(p \land q) \vdash \neg q \)

means as much as \( p, \diamond \neg(p \land q) \vdash \diamond \neg q \), or, by classical contraposition, \( p, \neg \diamond \neg q \vdash \neg \diamond \neg(p \land q) \); again, however, the conclusion \( (p \land q) \) follows from \( q \) only in conjunction with \( p \) and not just in conjunction with \( p \) simpliciter.

The question whether the law of consistency, CONSIS 1, also is dispensable for arbitrary negations or not, is much harder to decide. Several arguments might be given in favour of adopting \( \neg(p \land \neg p) \) as an essential theorem for arbitrary negations. First, CONSIS 1 is easily seen to follow from the subsequent weak version of tertium non datur:

\[(\text{TNDf})\quad \vdash (\neg p \lor \neg \neg p).\]

Since \( (p \land \neg p) \vdash p \) and \( (p \land \neg p) \vdash \neg p \), one obtains \( \neg p \vdash \neg(p \land \neg p) \) and \( \neg \neg p \vdash \neg(p \land \neg p) \) by CP 1, hence by OR \( (\neg p \lor \neg \neg p) \vdash \neg(p \land \neg p) \), and thus CONSIS 1 follows from TNDf. However, TNDf itself may be regarded as dispensable for arbitrary negations since it is not satisfied, e.g., by strong negations à la Def. 1. To wit, the assumption \( (\neg p \lor \neg \neg p) \) amounts to \( (\neg p \lor \neg \neg p) \), i.e. \( (\diamond p \supset \diamond p) \), and this principle holds only in \( S5 \) but not in other modal calculi.

Second, as was already pointed out earlier, unlike the classical formula \( (p \land \neg p) \), the conjunction \( (p \land \neg p) \) does not always represent a real contradiction. In particular \( (p \land \neg p) \) becomes perfectly self-consistent when \( \neg p \) is interpreted in the weak sense of Def. 2. Therefore one cannot expect it to be provably false in the classical sense, \( \neg(p \land \neg p) \). Still it seems plausible to require that \( (p \land \neg p) \) should be provably „false“ at least in the sense of the operator \( \neg \) as it occurs within the formula \( (p \land \neg p) \) itself. Yet this argument is not entirely conclusive either. After all, CONSIS 1 will not necessarily be satisfied by „weak“ negations à la Def. 2. To wit, \( \neg_{w}(p \land \neg_{w}p) \), „means“ as much as \( \diamond \neg(p \land \diamond \neg p) \), i.e. \( \diamond (p \supset p) \), and this formula, though holding in many modal logics, is not a theorem of each such logic.\(^{16}\) It might, however, be replied that the \( \diamond \) operator of such an „exotic“ modal system which does not satisfy \( \diamond (p \supset p) \) would better not be called a necessity operator and that the corresponding

\(^{16}\) In particular \( \diamond (p \supset p) \) can be falsified by the following Kripke-structure with three possible worlds i,j,k: Let the (irreflexive and non-transitive) accessibility relation \( R \) consist only of the pairs \( \langle i,j \rangle\), and \( \langle j,k \rangle\), and let \( p \) be true in \( j \) but false in \( k \). Then in \( j \) the proposition \( p \), and hence also \( (p \supset p) \) is false, and therefore \( \diamond (p \supset p) \) becomes false in world \( i \).
weak negation à la Def. 2 therefore doesn’t represent a serious counter-example to the validity of CONSIS 1 either.

Anyway, the issue of the dispensability or indispensability of CONSIS 1 need not be decided here once and forever. It will turn out below that CONSIS 1 is satisfied if not by each, then at least by the most prominent „types“ of negation operators, namely both by arbitrary weak and also by arbitrary strong negations. Since we are mainly interested in necessary conditions for negation operator, however, only entirely unproblematic principles shall be adopted as conditions of adequacy here. Therefore, foregoing CONSIS 1, we obtain the following condition which summarises the discussion of the preceding two sections:

(ADQ) Let \( L \) be a self-consistent logic such that not for every proposition \( q: \models_L q \). Then the unary operator \( \sim \) is a negation of \( L \) only if:

1. not for every proposition \( p: \models_L \sim p \);
2. for every proposition \( p: \) If \( \models_L q \), then \( \sim q \models_L \sim p \);
3. for every proposition \( p: \) If \( \models_L p \), then \( \models_L \sim\sim p \).

As was pointed out in connection with ADQ 3 above, it follows from (3) that whenever \( L \) is a „normal“ logic in the sense that, for at least one proposition \( p, \models_L p \), then also the subsequent two principles hold:

4. for at least one proposition \( p: \models_L \sim\sim p \)
5. for at least one proposition \( p: \models_L \sim p \).

In the subsequent section we will offer a coarse classification of the broad field of non-classical negation operators by examining some further principles which appear to be dispensable for arbitrary negations.

4 Weak and Strong Negation

In this section it is presupposed for convenience that the respective logics \( L \) contain the basic theory of conjunction and disjunction and that \( L \) is „normal“ in the sense that, for at least one proposition \( p, \models_L p \). In section 2, negation operators \( \sim \) have preliminarily been characterised as being weak or strong according to whether they fail to satisfy ex contradictorio quodlibet, ECQ, or tertium non datur, TND, respectively. Before redefining these notions in a slightly diverging way, let me first explain the intuitive idea behind this classification in some more detail. The most natural way of determining the „strength“ of a negation \( \sim \) would seem to proceed by way of comparison with classical negation \( \neg \): A negation \( \sim \) should evidently be considered as strong iff it is „at least as strong as \( \neg \)“ in the sense that \( \sim p \) logically entails \( \neg p \).
Thus, if ~ is a strong negation, then the conjunction \((p \land \neg p)\) logically entails the contradiction \((p \land \neg p)\) and must therefore itself be regarded as contradictory. In other words, if ~ is a strong negation, then p and \(\neg p\) can never be true together. Analogously ~ should be considered as weak iff ~ is „at most as strong as \(\neg \)“, i.e. if \(\neg p\) is logically entailed \(\neg p\). Accordingly, if ~ is a weak negation, then the disjunction \((p \lor \neg p)\) logically follows from the tautology \((p \lor \neg p)\) and hence is itself logically true. In other words, if ~ is a weak negation, then at least one of the propositions p and \(\neg p\) must always be true.

Unfortunately such comparisons between ~ and \(\neg\) will normally not be possible within the logic \(L\) itself since \(L\) usually does not contain \(\neg\) as an additional operator besides ~. One might, however, consider the „\(\neg\)-extension of \(L\)“, i.e. that system \(L^*\) which is obtained by adding the theory of classical negation to \(L\), and then define ~ to be strong iff \(\neg p \vdash_{L^*} \neg p\). Similarly, ~ might be defined to be weak iff \(\neg p \vdash_{L^*} \neg p\). If this approach were chosen, the following basic facts could be proved:

- If the negation ~ of a logic \(L\) satisfies ECQ, then ~ is strong.
- If the negation ~ of a logic \(L\) satisfies TND, then ~ is weak.\(^{17}\)

Furthermore, from an intuitive point of view, the negation ~ should also conversely come out as strong only if it satisfies ECQ and correspondingly as weak only if it satisfies TND. However, the latter relations could seldom or never be proved because we have no guarantee that the deductive relations which hold in the \(\neg\)-extension \(L^*\) already hold in \(L\) itself. To be sure, if ~ is a weak negation, then in \(L^*\) \((\neg p \supset \neg p)\) becomes provable, and hence also \(\vdash_{L^*} (p \lor \neg p)\). But evidently there is no way to infer that \((p \lor \neg p)\) is already a theorem of \(L\)!

Similarly, if ~ is strong, then \(\neg p \vdash_{L} \neg p\) and hence by the laws of conjunction \((p \land \neg p) \vdash_{L} q\). But how could we ever derive from this that already in \(L\) \((p \land \neg p) \vdash_{L} q\)? In view of these difficulties one better drops the \(L^*\)-approach and adopts instead the satisfaction of ECQ and of TND as defining marks of strong and weak negations:

**Def. 5** The negation-operator ~ of a logic \(L\) is a strong negation :\(\iff\)
\[(p \land \neg p)\] represents a contradiction in \(L\), i.e. \(p, \neg p \vdash_{L} q\).

\(^{17}\) Proof: (1) If ECQ holds in \(L\), i.e. if \((p \land \neg p) \vdash_{L} q\), then in \(L^*\) \((p \land \neg p) \vdash_{L^*} (p \land \neg p)\) from which one obtains by CP \(1\) \(\neg(p \land \neg p) \vdash_{L^*} \neg(p \land \neg p)\). But, classically, one also has \(\vdash_{L^*} \neg(p \land \neg p)\), so that in sum \(\vdash_{L^*} \neg(p \land \neg p)\), i.e. \(\vdash_{L^*} (p \lor \neg p)\), or \(\neg p \vdash_{L^*} \neg p\). (2) Since \(L^*\) contains the theory of classical negation, the following instance of disjunctive syllogism, \(\neg p, (p \lor \neg p) \vdash_{L^*} \neg p\), holds; hence, if TND is provable in \(L\), then \(\neg p \vdash_{L^*} \neg p\).
The negation-operator ~ of a logic L is a weak negation if and only if (p ∨ ~p) represents a tautology in L, i.e. \( \vdash (p \lor \neg p) \).

The „weakness“ and „strongness“ of a negation can alternatively be characterised as follows:

**Theorem 1:**

1.1 A negation ~ is weak if and only if it satisfies:
\[
\text{CONSIS 3} \quad \text{If } \neg p \vdash p, \text{ then } \vdash p;
\]

1.2 A negation ~ is strong if and only if it satisfies:
\[
\text{CONSIS 4} \quad \text{If } p \vdash \neg p, \text{ then } p \vdash q.
\]

Proofs of this theorem and of all subsequent theorems are given in the Appendix.

Now if a negation-operator of a logic L is both weak and strong, then it is classical either in the sense that, in the „¬-extension“ \( L^* \) of L, \( \neg p \) is deductively equivalent to \( \neg \neg p \), or in the sense that, in L itself, ~ satisfies the characteristic axioms of classical negation:

(Def. 7) The negation-operator ~ of a logic L is a classical negation if and only if:
\[
\vdash (p \lor \neg p) \text{ and } p, \neg p \vdash q.
\]

Accordingly a negation is genuinely strong iff it is strong but not classical; and ~ is genuinely weak iff it is weak but not classical. Let us now see how these properties are related to the validity of certain other axioms and rules of deduction besides ECQ and TND. To begin with, remember that according to our condition of adequacy ADQ every negation-operator ~ of a logic L (which contains the basic theory AND and OR) satisfies in particular the following de Morgan principles:

\[
\begin{align*}
\text{DM 1} \quad &\quad (\neg p \lor \neg q) \vdash \neg (p \land q) \\
\text{DM 2} \quad &\quad \neg (p \lor q) \vdash \neg (p \land \neg q).
\end{align*}
\]

Next we find that the „law of consistency“ – which according to Gabbay is indispensable for arbitrary negations – is satisfied at least both by arbitrary weak and by arbitrary strong negations:

**Theorem 2:**

2.1 Every weak negation as well as every strong negation satisfies:
\[
\text{CONSIS 1} \quad \vdash \neg (p \land \neg p)\]

---

\[18\] DM 1 and DM 2 appear to be more or less universally accepted in the literature on non-classical negation. In particular they hold in the weak „direkte Aussagenlogik“ of [Kutschera 1985]. Also [Pearce/Wagner 1991: 442] who otherwise reject a stronger version of contraposition accept at least the inference from \( \neg p \) (or from \( \neg q \)) to \( \neg (p \land q) \).

\[19\] As the reader may easily verify, CONSIS 1 is deductively equivalent to the following variants:
\[
\begin{align*}
\text{CONSIS 1a} \quad &\quad \text{If } p \vdash \neg p, \text{ then } \vdash \neg p.
\end{align*}
\]
2.2 This principle in turn entails
\text{CONSIS 2} \quad \text{If } \neg p \vdash p, \text{ then } \vdash \neg \neg p.

2.3 The latter principle is deductively equivalent to doubly negated TND:
\text{DNTND} \quad \vdash \neg \neg (p \lor \neg p).

These principles may thus be considered as „almost universally valid“. Therefore we want to
call the negation of a logic \( L \) \textit{almost classical} iff it becomes classical as soon as CONSIS 1
holds in \( L \):

(Def. 8) The negation-operator \( \neg \) of a logic \( L \) is \textit{almost} classical: ⇔
\begin{align*}
& \text{If } \vdash_L \neg (p \land \neg p), \text{ then } \neg \text{ becomes both weak and strong, i.e. } \vdash_L (p \lor \neg p) \text{ and } \vdash_L p, \neg p. \\
& \text{Next let us consider the laws of double negation the validity of which is largely independent}
& \text{of the „strongness“ or „weakness“ of } \neg.^{20}
\end{align*}

Theorem 3:

3.1 A negation \( \neg \) satisfies DNI iff it satisfies one of the subsequent principles:
\begin{align*}
& \text{DM 3} \quad (p \land q) \vdash \neg (\neg p \lor \neg q), \\
& \text{DM 4} \quad (p \lor q) \vdash \neg (\neg p \land \neg q), \\
& \text{CP 6} \quad \text{If } p \vdash \neg q, \text{ then } q \vdash \neg p.
\end{align*}

3.2 A negation \( \neg \) satisfies DNE iff it satisfies one of the subsequent principles:
\begin{align*}
& \text{DM 5} \quad \neg (\neg p \lor \neg q) \vdash (p \land q), \\
& \text{DM 6} \quad \neg (\neg p \land \neg q) \vdash (p \lor q), \\
& \text{CP 5} \quad \text{If } \neg p \vdash q, \text{ then } \neg q \vdash \neg p.
\end{align*}

3.3 If a negation satisfies DNI then it also satisfies:
\begin{align*}
& \text{DM 7} \quad (\neg p \land \neg q) \vdash \neg (p \lor q).
\end{align*}

3.4 If a negation satisfies DNE then it also satisfies:
\begin{align*}
& \text{DM 8} \quad (p \land q) \vdash \neg (\neg p \lor \neg q).
\end{align*}

Some major results may now be summarised as follows:

Theorem 4:

4.1 A negation \( \neg \) is strong whenever it satisfies disjunctive syllogism:
\begin{align*}
& \text{DISSYL} \quad (p \lor q), \neg p \vdash q.
\end{align*}

4.2 A negation \( \neg \) satisfies DNI whenever it satisfies conjunctive syllogism:
\begin{align*}
& \text{CONSYL} \quad p, \neg (p \land q) \vdash \neg q;
\end{align*}

4.3 A negation \( \neg \) is almost classical whenever it satisfies strong contraposition:
\begin{align*}
& \text{CP 7} \quad \text{If } \neg p \vdash \neg q, \text{ then } q \vdash \neg p.
\end{align*}

4.4 A negation \( \neg \) is almost classical if it satisfies both DNE and DNI;

\begin{align*}
& \text{CONSIS 1b} \quad \text{If } p \vdash \neg q \text{ and } p \vdash \neg q, \text{ then } \vdash \neg p.
\end{align*}

\(^20\) In intuitionistic logic, however, DNE is equivalent to TND. Thus TND follows from DNE in
conjunction with the intuitionistically valid DNTND; and theorem 4.45 of [Heyting 1930] shows
that DNE conversely follows from TND.
4.5 A weak negation becomes classical if it satisfies DISSYL;
4.6 A strong negation becomes classical if it satisfies DNE.

In the subsequent section some of these results will be applied to the negation operators of diverse systems of paraconsistent logic.

5 Paraconsistent Logic

The basic idea of paraconsistent logic is to allow an „inconsistent“ pair of propositions \{p, ~p\} to be true without thereby trivialising the logic, i.e. without thereby having to accept every other proposition q as true, too. In the words of [Priest/Routley 1984: 3]: Let \(\Sigma\) be a „set of sentences. \(\Sigma\) is inconsistent iff, for some A, \{A, ~A\} \in\Sigma. \(\Sigma\) is trivial iff for all B, B \in \Sigma. The important fact about paraconsistent logics is that they provide the basis for inconsistent but non-trivial theories.“ Thus any paraconsistent logic has to dispense with ECQ, or in other words:

**Corollary 1**: Paraconsistent negation never is a strong negation.

For a closer characterisation of paraconsistent negation, three different types of paraconsistent logics must be distinguished which have been dubbed by [Priest/Routley 1984] „The non-adjunctive approach“, „The positive logic plus approach“, and „The relevant approach“.

5.1 The „non-adjunctive“ approach

The first approach originates from [Jaskowski 1948/1969] who presented a „discussive propositional calculus“, \(\mathbf{J}\), in which the usual rule of „adjunction“ (or conjunction), i.e. \(p, q \Rightarrow (p \land q)\), is no longer unrestrictedly valid although, on the other hand, the system contains the full axiomatic counterpart of the standard theory of conjunction\(^{21}\). Neglecting this peculiarity of system \(\mathbf{J}\), let it be noted that in \(\mathbf{J}\) instead of the fundamental principle CP 1 only certain special instances of contraposition such as DM 2 are postulated as axioms. Moreover, not even (EQUI): If \(p \limp q\), then \(~p \limp ~q\) appears to hold unrestrictedly. \(\mathbf{J}\) contains only a small choice of special instances of this indispensable rule\(^{22}\), and it seems very doubtful whether these suffice to derive EQUI. Anyway it is easy to show

\(^{21}\) The failure of the rule of conjunction is due to the fact that the system does not provide for ordinary modus ponens; \(\mathbf{J}\) only has modus ponens for theorems: If \(\vdash p\) and \(\vdash (p \limp q)\), then \(\vdash q\).

\(^{22}\) To wit, the following axioms as given in the reconstruction by [Achtelik et al. 1981: 3-4]:

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Theorem 5: The „negation“ operator of Jaskowski’s logic J – to be denoted as \(~J\) – either fails to satisfy CP 1 (and hence, according to ADQ, is not a real negation) or it collapses into classical negation.

A formal proof may be found in the Appendix. Here only a brief analysis of the situation shall be given which shows why at any rate \(~J\) cannot be a paraconsistent negation operator. System J contains the full theory of double negation, in particular principle DNI. Furthermore it contains the following axiom „A3“: \(~J(p \lor \neg Jp) \supset q\). Now if \(~J\) were a real negation operator, then according to Theorem 3.3 besides DNI also DM 7, \((\neg Jp \land \neg Jq) \vdash \neg J(p \land q)\), would hold in J. But Jaskowski’s „A3“ in conjunction with DM 7 immediately entails ECQ. Thus if \(~J\) really were a negation, then it would be a strong negation and hence (Corollary 1) no paraconsistent negation at all.

5.2 The „positive logic plus“ approach

According to [Priest/Routley 1984: 6], the second approach to paraconsistent logic adopts „the whole of the positive logic standard but merely allow[s] for a non-classical behaviour of negation.“ In what follows, three main variants of this approach shall be considered, namely the systems J1 - J5 of [Arruda/da Costa 1970], the various „C-systems“ of [da Costa 1974], and the „Dialectical Logic“ DL of [da Costa/Wolf 1980]. Let us investigate these systems in chronological order and consider the Ji (1 ≤ i ≤ 5) first. As was pointed out by [Bunder 1983: 43], despite the fact that the Ji do not have modus ponens, they „have all the theorems of positive intuitionistic logic.“ Now, the negation-operators of all these systems – to be denoted by \(~ ji\) – satisfy TND, DNE + DNI, and CONSIS 1. So if any of the \(~ ji\) would be a negation at all, then, since it satisfies DNE plus DNI, it would be „almost classical“ (Theorem 4.4); hence, since \(~ ji\) also satisfies CONSIS 1, it would have to be classical (Def. 8)!

Actually, however, most of the Ji-“negations“ are no negations at all, since they fail to satisfy the basic principle of contraposition, CP 1. Thus, e.g., the J2-“negation“-operator, though satisfying the dispensable principle DM 7, \((\neg Jp \land \neg Jq) \vdash \neg J(p \lor q)\), fails to satisfy the converse DM 2, \((\neg Jp \lor \neg Jq) \vdash \neg J(p \land q)\), which follows from CP 1 and hence is indispensable for any real negation. Similarly, although the stronger system J3 contains two further axioms, \(~ Jp, q \vdash \neg J(p \land q)\) and \(p, \neg q \vdash \neg J(p \land q)\), from which \((\neg Jp \land \neg Jq) \lor (p \land \neg Jq) \vdash \neg J(p \land q)\) may be derived, the latter principle is still weaker than the other indispensable principle DM 1,

\begin{align*}
\text{„A4} & \quad \neg J(p \lor q) \supset \neg J(q \lor p) \ldots \\
\text{A6} & \quad \neg J(\neg Jp \lor q) \supset \neg J(p \lor q) \ldots \\
\text{A8} & \quad \neg J((p \lor q) \lor r) \supset \neg J(p \lor (q \lor r)).
\end{align*}
Only the systems $J_4$ and $J_5$ contain CP 1 in an (almost) unrestricted form, viz. in form of the rule(s): If $\Delta, p \vdash q$ and $\Delta \vdash \lnot q$ then $\Delta \vdash \lnot p$ (where, in the case of $J_4$, $\Delta$ must be empty).

Next let us turn to the negation operators of the „$C$-systems“ ($C_1, \ldots C_n, \ldots C_\omega$) of [da Costa 1974] which according to [Priest/Routley 1989: 176] represent conservative extensions of positive intuitionistic logic. In contrast to intuitionistic negation, however (and also in contrast to the „dialectical“ approach to be discussed below), the $C$-negations – to be symbolised as $\lnot_C$ – do satisfy TND. Therefore it follows by Def. 6 that if the negation operators of these $C$-systems really are negations, then they have to be weak negations. Moreover, according to da Costa’s axiomatisation, $\lnot_C$ satisfies DNE but fails to satisfy DNI. Since intuitionistic negation conversely satisfies DNI but fails to satisfy DNE, it might seem plausible to follow [Priest/Routley 1989: 176;183] in characterising $\lnot_C$ as kind of an „anti-intuitionistic“ negation. However, [da Costa 1974: 493] further postulated that „the principle of contradiction, $\lnot(p \land \lnot p)$, must not be a valid schema“ of paraconsistent logic. Taken literally, this means that the $C$-systems do not satisfy CONSIS 1. According to Theorem 2.1, then, $\lnot_C$ could be neither weak nor strong. But it was argued above that if $\lnot_C$ would be a real negation at all then it would be a weak negation. Hence $\lnot_C$ cannot fulfil the relevant conditions of adequacy, ADQ. As a matter of fact, [Priest/Routley 1989: 165] noticed that da Costa’s consequence relation fails to satisfy some elementary instances of CP 1 such as $\lnot p \vdash_C \lnot(p \land q)$ or $\lnot(p \lor q) \vdash_C \lnot p$. Thus:

**Corollary 2** The „negation“ operator of da Costa’s $C$-systems, $\lnot_C$, does not constitute a real negation since it does not satisfy CP 1.

Let it be noted in passing that the above quoted requirement according to which the „principle of contradiction“ must not be a theorem of paraconsistent logic, does not necessarily mean to give up CONSIS 1. To be sure, in a language containing just one negation operator, $\lnot$, the usual version of the „principle of (non-) contradiction“, $\vdash_C \lnot(p \land \lnot p)$, cannot be distinguished from its counterpart

(\textbf{CONSIS 1*}) $\vdash_C \lnot(p \land \lnot p)$.

The latter says that the two „contradictory“ propositions $p$ and $\lnot p$ are logically incompatible in the sense of classical $\text{PC}$, or that the conjunction $(p \land \lnot p)$ represents a classical contradiction.

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23 This appears to accord quite well with the characterization of $\lnot_C$ as a „weak negation“ in [Alves 1984: 19]. Actually, however, Alves had strengthened da Costa’s negation by taking also double negation introduction as an additional axiom.
As was already stressed in section 3 above, CONSIS 1* need not generally hold for arbitrary negation operators. In particular, if ~ represents some form of *weak* negation, then (p ∧ ~p) may perfectly well be self-consistent. Anyway paraconsistent logic seems to be fairly compatible with the validity of CONSIS 1; the only thing that paraconsistent logicians have to insist on is that the related principle CONSIS 1* should not come out as valid. This, however, seems already warranted by da Costa’s second requirement according to which from „two contradictory formulas, p and ~p, it will not in general be possible to deduce an arbitrary formula q“.

Third, consider the „dialectical“ approach of [da Costa/Wolf 1980] The negation operator of this system – to be symbolised as ~\_DL – appears to be neither weak nor strong, since it is postulated (o.c., p.196) that the „laws of excluded middle (p ∨ ~\_DLp) and of non-contradiction ~\_DL(p ∧ ~\_DLp) must be non-theorems“. On the other hand, ~\_DL is supposed to satisfy de-Morgan principles DM 1, 2, 7 and 8. This suffices to conclude:

**Theorem 6:** The „negation“ operator ~\_DL of da Costa/Wolf’s „dialectical logic“ DL either fails to satisfy CP 1 (and hence is not a real negation) or ~\_DL collapses into classical negation.

The following analysis is meant to show what has gone wrong. Da Costa/Wolf want to distinguish between a „paraconsistent“ proposition α and a „well-behaved“ proposition α\_0 by the fact that only the latter but not the former satisfies the principle of non-contradiction, CONSIS 1. Accordingly they define a „well-behaved“ proposition α\_0 as one for which ~\_DL(α ∧ ~\_DLα), holds, and they adopt certain axioms which warrant that in the case of such „well-behaved“ propositions the theorems of ordinary, classical PC become valid. In particular, as was shown by [Alves 1984: 19], within the framework of system DL one may introduce a „strong negation“

**Def. 9**  
\[ \sim^*\alpha =_{df} \sim\_DL\alpha \land \sim\_DL(\alpha \land \sim\_DL\alpha), \]

which can be proven to possess „all the properties of classical negation“. Now it is easy to see that if da Costa/Wolf’s „weak“ negation ~\_DL were a negation at all, then it would be deductively equivalent to „strong“ ~*. For, on the one hand, ~\_DL by Def. 9 trivially entails ~\_DLα; conversely, the simple conjunction law (α ∧ ~\_DLα) ├ α by way of CP 1 entails that

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\[ \neg_{DL} \alpha \vdash \neg_{DL} (\alpha \land \neg_{DL} \alpha); \text{ hence one trivially obtains also } \neg_{DL} \alpha \vdash \neg_{DL} \alpha \land \neg_{DL} (\alpha \land \neg_{DL} \alpha), \]
i.e. \[ \neg_{DL} \alpha \vdash \neg^* \alpha! \]

Summing up, then, it turns out that most variants of the „positive logic plus“-approach involve the use of a „negation“ operator which fails to satisfy CP 1 and hence is no real negation at all. The (implicit or explicit) rejection of CP 1 is motivated by the following fact. Since all these systems contain the theorems of the so-called positive calculus, they have in particular the standard implication principle \[ \vdash (p \supset (q \supset p)) \] as a theorem. As was pointed out by [Priest/Routley 1989: 177], any such system therefore has to reject contraposition in the form

\[ (\text{CP 2}) \quad \vdash (q \supset p) \supset (\neg p \supset \neg q), \]

because otherwise \((p \supset (\neg p \supset q))\), i.e. the paraconsistently invalid ECQ, would become provable. As was already remarked in section 2 above, however, this does not necessarily mean that CP 1 has to be given up. It would suffice just to drop axiom CP 2 (or, equivalently, its counterpart CP 3).

### 5.3 The „relevant“ approach

The third and last class of paraconsistent logics to be considered here has been developed by [Routley/Meyer 1977]; it represents a subclass of relevance logic. It is somewhat difficult to evaluate this approach within the framework of the present paper since relevance logic in general rejects certain laws of the „positive“ propositional calculus so that the corresponding consequence relation does not fully satisfy the requirement MONO stated in section 1 above. Therefore the relevant results of Theorems 1 - 4 (section 3) do not automatically apply to these systems. Yet at least the following rough (and hypothetical) evaluation of two particular calculi \(F\) and \(BF\) is possible.

The negation of the „basic negative system \(F\)“ – to be denoted by \(\neg_F\) – is referred to by Routley/Meyer as „De Morgan minimal negation“*. This name seems very appropriate because \(\neg_F\) satisfies not only the de Morgan laws DM 1, 2, 7, and 8, but also the basic principle CP 1. Hence \(\neg_F\) satisfies the most important condition of adequacy ADQ 1. Furthermore also the „non-triviality“-condition ADQ 2 appears to hold in \(F\). However, as will become clear soon, \(\text{CONSIS 1}\) cannot be a theorem of \(F\) so that it follows by Theorem 2.1:

**Corollary 3:** If the negation-operator \(\neg_F\) of Routley/Meyer’s relevance logic \(F\) is a real negation, then it is neither weak nor strong.
On the other hand there is strong reason to believe that the premise of this Corollary is false. In other words, \( \sim_F \) most likely is not a real negation, after all. For, reflecting upon the axioms and rules of deduction of \( F \), it seems impossible to derive, for any theorem \( \alpha \) of \( F \), also \( \sim_F \sim_F \alpha \) as a theorem. Hence ADQ 4 apparently is violated.

In the stronger system \( BF \) which is obtained by adding to \( F \) both double negation introduction and double negation elimination, however, ADQ 4 is trivially satisfied. The resulting negation, \( \sim_{BF} \), is referred to by Routley/Meyer as a „normal negation“. According to Theorem 4.4 every negation satisfying both DNI and DNE is „almost classical“. On the other hand, the authors proved (by semantical means) that \( \sim_{BF} \) is not classical negation. Therefore it follows by Def. 8 that \( \sim_{BF} – \) and a fortiori also \( \sim_F – \) cannot satisfy CONSIS 1! Thus (again by Theorem 2.1) we obtain:

**Corollary 4:** If the negation-operator \( \sim_{BF} \) is a real negation, then it is neither weak nor strong.

As [Routley/Meyer 1977: 61] mention, \( \sim_{BF} \) may „readily be strengthened to a classical negation by adding [...\] D2. \( p \land (\sim p \lor q) \rightarrow q.\)“ As a matter of fact, in view of the trivial law of disjunction, \( \sim p \rightarrow (\sim p \lor q) \), D2 immediately entails ECQ. \( p \land \sim p \rightarrow q.\) Thus any „normal“ negation which satisfies D2 is strong and hence (by Theorem 4.6) must be classical since it also satisfies DNE. Summing up the results of the foregoing investigation, then, it seems that \( \sim_{BF} \) is the only serious candidate for a genuine paraconsistent negation.

**5.4 Some further operators in the vicinity of paraconsistent negation**

To conclude our discussion let us consider the investigations of [Bunder 1984] who introduces certain „negation“ operators by the scheme:

**Def. 10** \( \sim_Bp \equiv \text{df} \ (p \Rightarrow X). \)

Here \( X \) is a variable either for a certain constant proposition or for a higher-order-sentence involving propositional quantifiers, and \( \Rightarrow \) may apparently be interpreted either as material implication or as strict implication. Without specifying a particular sentence \( X \), any such „negation“ may be shown to satisfy DNI, CONSIS 1, and (besides some other less interesting principles) also CP 2 and CP 6. Moreover, the particular choice of \( X = (p \supset p) \) yields a „negation“ \( \sim_{B_1} \) such that, for every proposition \( p \), \( \sim_{B_1}p \) becomes provable. This clearly violates our „non-triviality“-condition ADQ 1, hence:
Corollary 5: Bunder’s negation operator $\sim_{B1} p$ fails to satisfy ADQ 1 and thus is no real negation.

Another choice of $X = (\forall p)p$ leads to a „negation“ $\sim_{B2}$ which satisfies ECQ and thus represents a (non-paraconsistent) strong negation. However, this strong negation is not very interesting and new. As [Bunder 1984: 78] himself points out, the addition of $\sim_{B2}$ to the underlying calculus of positive logic yields nothing else but „a full intuitionistic system“, i.e.:

Corollary 6: Bunder’s negation operator $\sim_{B2}$ is a strong (intuitionistic) negation and thus no paraconsistent negation.

Finally, consider the „strong“ negation introduced in [Nelson 1949] and discussed, e.g., in [Goranko 1985]. „Nelson-negation“ – to be denoted as $\sim_N$ – is sometimes also referred to as „superintuitionistic“ negation because it entails intuitionistic negation. Furthermore, since $\sim_N$ satisfies axiom ECQ, it might appear to really deserve the name of a „strong negation“. As a matter of fact, however:

Corollary 7: Nelson’s „strong negation“ $\sim_N$ is no real negation at all, since it does not satisfy CP 1.

Actually, as is evident from „Exercise 7“ in [Gabbay 1981: 124], it does not even satisfy the more elementary principle EQUI. Moreover, if $\sim_N$ would be turned into a real negation by adopting CP 1 as an additional axiom, then the result wouldn’t yield any interesting new negation, but just classical negation. Since on the one hand $\sim_N$ is supposed to satisfy ECQ, it would have to be a strong negation; on the other hand $\sim_N$ is also supposed to satisfy double-negation elimination DNE; thus – according to Theorem 4.6 – it would be classical.

6 Appendix: Proofs

Theorem 1:

1.1 (a) If $\sim p \vdash p$, then because of $p \vdash p$ by OR $(p \lor \sim p) \vdash p$; hence TND entails the desired conclusion $\vdash p$. (b) According to DM 2 $\sim(p \lor \sim p) \vdash \sim p$, hence by OR also $\sim(p \lor \sim p) \vdash (p \lor \sim p)$, from which one infers by means of CONSIS 3 that $\vdash (p \lor \sim p)$.

1.2 (a) If $p \vdash \sim p$, then because of $p \vdash p$ by AND $p \vdash (p \land \sim p)$; hence by means of ECQ $p \vdash q$. (b) By DM 1 $\sim p \vdash \sim (p \land \sim p)$, a fortiori $(p \land \sim p) \vdash \sim (p \land \sim p)$, so that it follows by CONSIS 4 that $(p \land \sim p) \vdash q$, i.e. $p, \sim p \vdash q$.

Theorem 2:
2.1 (a) As a matter of fact, we prove the somewhat stronger result that CONSIS 1 will already be satisfied by any „semi-weak“ negation \( \sim \), which instead of TND satisfies only the weakened principle TNDF: \( \sim (p \lor \sim p) \). Since by DM 1 both \( \sim p \models \sim (p \land \sim p) \) and \( \sim p \models \sim (p \land \sim p) \), it follows by OR and by TNDF that \( \models \sim (p \land \sim p) \). (b) Again we prove the slightly stronger result that CONSIS 1 is satisfied by any „semi-strong“ negation \( \sim \), which instead of ECQ satisfies only the weakened principle ECQF: \( (p \land \sim p) \models \sim q \). From \( (p \land \sim p) \models \sim q \) one obtains by CP 1 \( \sim q \models \sim q \). But L is supposed to be a normal logic so that there exists at least one \( q \) such that \( \models q \); in view of ADQ 4 it follows that \( \models \sim q \); thus we obtain \( \models \sim (p \land \sim p) \).

2.2 CONSIS 2 follows from CONSIS 1 because, if \( \sim p \models p \), then by elementary contraposition \( \sim p \models \sim p \) so that by CONSIS 1a \( \models \sim p \).

2.3 (a) Since by DM 2 \( \sim (p \lor \sim p) \models \sim p \) and by OR \( \sim p \models (p \lor \sim p) \), one obtains \( \sim (p \lor \sim p) \models (p \lor \sim p) \); hence DNTND follows from CONSIS 2. (b) Conversely, CONSIS 2 is obtained from \( \models \sim (p \lor \sim p) \) as follows: If \( \sim p \models p \), then because of the trivial \( p \models p \) and by OR also \( (p \lor \sim p) \models p \), hence by double application of CP we obtain \( \sim (p \lor \sim p) \models \sim p \) so that the conclusion of CONSIS 2 follows by DNTND.

Theorem 3:

3.1 (a) DNI entails DM 3: By DM 1 we have \( \sim (p \lor \sim q) \models \sim (p \land q) \), hence by CP 1 \( \sim (p \lor \sim q) \models \sim (p \land q) \) and by DNI \( (p \lor q) \models \sim (p \land q) \). (b) To see that DM 3 entails DNI, just set \( p = q \). (c) DNI entails DM 4: Again by DM 1 we have \( \sim (p \lor \sim q) \models \sim (p \land q) \), hence by DNI also \( (p \lor q) \models \sim (p \land q) \). (d) Conversely DNI is obtained from DM 4 by setting \( q = p \). (e) DNI entails CP 6: If \( p \models q \), then by CP 1 \( \sim q \models \sim p \), hence by DNI also \( q \models \sim p \). (f) Conversely DNI follows from CP 6 by the substitution \( p/q; q/p! \).

Note, incidentally, that Došen (1988: 382-3) already proved CP 6 to be deductively equivalent (within the framework of what he calls Hilbert system BC) to CP 1 plus DNI.

3.2 (a) DNE entails DM 5: By DM 2 \( \sim (p \lor \sim q) \models \sim (p \lor \sim q) \), hence by DNE also \( \sim (p \lor \sim q) \models (p \land q) \). (b) Conversely DNE is obtained by DM 5 by just setting \( p = q \). (c) DNE entails DM 6: Again by DM 2 we have \( \sim (p \lor q) \models \sim (p \land q) \), hence by CP 1 \( \sim (p \lor \sim q) \models \sim (p \lor q) \) and thus by DNE \( \sim (p \lor q) \models (p \lor q) \). (d) Conversely DM 6 entails DNE by the substitution \( q/p \). (e) DNE entails CP 5: If \( \sim p \models q \), then by CP 1 \( \sim q \models \sim q \), hence by DNE \( \sim q \models p \). (f) Conversely DNE follows from CP 5 by the substitution \( p/q; q/p! \).
3.3 \((p \lor q) \vdash (\neg p \lor \neg q)\) by DNI and \((\neg p \lor \neg q) \vdash (\neg p \land \neg q)\) by DM 1; hence by CP 1 \((\neg p \land \neg q) \vdash (p \lor q)\) from which one obtains DM 7 by another application of DNI.

3.4 Similarly, \((\neg p \lor \neg q) \vdash (\neg p \land \neg q)\) and \((\neg p \land \neg q) \vdash (p \land q)\) by DM 2 and DNE; hence by CP 1 \((p \land q) \vdash \neg (p \lor q)\) from which DM 8 follows by another application of DNE.

Theorem 4:

4.1 Since trivially \(p \vdash (p \lor q)\), the premises of ECQ, \(\{p, \neg p\}\), immediately entail the premises of DISSYL, \(\{p \lor q, \neg p\}\), so that one obtains the desired conclusion \(p, \neg p \vdash q\).

4.2 CONSYL implies ECQF because in view of the trivial \(\neg p \vdash (p \land q)\), the premises of ECQF, \(\{p, \neg p\}\), entail the premises of CONSYL, \(\{p, \neg (p \land q)\}\), so that one obtains in sum \(p, \neg p \vdash \neg q\). Now, if in CONSYL \(\neg p\) is substituted for \(q\), one obtains \(p, \neg (p \land \neg p) \vdash \neg \neg p\); but, according to the proof of the strengthening of Theorem 2.1 given above, ECQF already entails CONSIS 1: \(\vdash \neg (p \land \neg p)\); hence in sum \(p \vdash \neg \neg p\).

4.3 Let \(\neg\) satisfy CP 7; then, if also \(\neg (p \land \neg p)\), both ECQ and TND become provable. For, on the one hand, \(\vdash \neg (p \land \neg p)\) immediately yields \(\neg q \vdash \neg (p \land \neg p)\), hence by CP 7 \((p \land \neg p) \vdash q\), i.e. ECQ. On the other hand by DM 2 one has \(\neg (p \lor \neg p) \vdash \neg p \land \neg q\); by ECQ also \((\neg p \land \neg q) \vdash \neg \neg (p \land \neg p), hence by transitivity \((p \lor \neg p) \vdash \neg \neg (p \land \neg p), from which one obtains by CP 7 that \((p \land \neg p) \vdash \neg (p \lor \neg p)\); but it is presupposed that \(\vdash \neg (p \land \neg p)\); hence \(\vdash (p \lor \neg p)\), i.e. TND.

4.4 Almost trivially DNE and DNI (in conjunction with CP 1) entail the strong contraposition principle CP 7 so that by Theorem 4.3 the negation is almost classical.

4.5 If a weak negation \(\neg\) satisfies DISSYL, then according to Theorem 4.1 it is also strong and hence classical.

4.6 If \(\neg\) is a strong negation, then by Theorems 2.1–2.3 it satisfies DNTND: \(\vdash \neg \neg (p \lor \neg p)\); so if \(\neg\) also satisfies DNE, one immediately obtains \(\vdash (p \lor \neg p)\), i.e. \(\neg\) is also weak and hence classical.

Theorem 5:

On the one hand, since \(\neg\) satisfies both DNI and DNE, it is almost classical (Theorem 4.3). On the other hand, if \(\neg\) is a real negation, then it has to satisfy CONSIS 1 since Jaskowski’s axiom „A3“, i.e. \(\neg p \vee \neg q \supset \neg q\), in conjunction with CP 1 is easily seen to entail DNTND: Consider any theorem \(\alpha\); because of DNI also \(\neg \neg j \alpha \supset \alpha\) is a theorem; but according to „A3“ one has in particular \(\neg j (p \lor \neg p) \supset \neg j \alpha\); so if CP 1 would hold, we would obtain \(\neg j \neg j \alpha \supset \neg j \neg j (p \lor
\(\neg p\), and hence DNTND would become provable. According to the strengthening of Theorem 2.1 this already suffices to conclude that CONSIS 1 holds; hence (by Def. 9) \(\sim_1\) would be classical.
6 Literature

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