Wolfgang Lenzen

Epistemic Logic

0. Introduction

0.1 History of epistemic logic

The core meaning of the Greek word *episteme* is *knowledge*. Thus, taken literally, epistemic logic represents the logic of knowledge. In modern philosophy, however, *epistemic logic* is used as a technical term not only for the logic of knowledge but also for the logic of belief, (although the latter might more appropriately be referred to as *doxastic logic* from the Greek *doxa* to mean *belief*).

Like logic in general, also epistemic logic in particular may be said to have been founded by Aristotle. This is true at least in the sense that several passages in *De Sophisticis elenchis* and in the *Prior* and *Posterior Analytics* deal with basic issues of what is nowadays conceived of as epistemic logic. More detailed investigations of principles of epistemic logic may be found in the manuals of Medieval authors such as Buridanus, Burleigh, Ockham, and Duns Scotus (cf., e.g., Chisholm 1963 and Boh 1986). However, systematic calculi of epistemic logic have only been developed after the elaboration of possible-worlds-semantics in the mid of our century. The most important works to be mentioned here comprise Carnap 1947, Kripke 1959, and Hintikka's pioneering *Knowledge and Belief* of 1962. Further steps towards the establishment of epistemic logic as a particular branch of modal logic have been taken by Kutschera 1976 and by Lenzen 1980a.

What is common to these approaches is that they remain *static* in character, i.e. they only describe the "logical" structure of the belief- or knowledge-system of a certain subject *a* at a certain time *t*. The basic principles for the *dynamics* of epistemic systems have been investigated esp. by Gärdenfors 1988 (cf., e.g., the contribution "Revision of belief systems" in section C III of this Handbook). Another generalization of epistemic logic has recently been attempted in the field of computer science (cf. Fagin et al. 1994) where one tries to model in particular the effects of communication between *n* subjects a_i for the joint knowledge of a "distributed system" $S=\{a_1,...,a_n\}$. Such considerations, however, fall outside the scope of this paper which only aims at describing, in barest outlines, the basic laws for propositional logics of *belief*, *knowledge*, and *conviction* and at discussing some selected issues related to "quantifying in" epistemic contexts.

0.2 Methodology of epistemic logic

Although epistemic logic exists as a branch of philosophical logic for quite a long time, it remains to be explained in which sense of the word *logic* epistemic logic constitutes a logic at all, or – to put it in the form of the sceptical question of Hocutt 1972 – "Is epistemic logic possible?". The general problem behind this question may be illustrated as follows. Take any propositional attitude, $\phi(a,p)$, which a certain subject a bears towards a proposition (or a state of affairs expressed by the proposition) p; let another proposition q be logically equivalent to p, $\vdash p \leftrightarrow q$. Then there appears to be no "logical" guarantee that a bears the same attitude ϕ also towards proposition q, for it seems always possible that a does not "see" (and hence doesn't know) that p and q are logically equivalent. Thus, in a certain sense, the following situation always seems possible: $\vdash p \leftrightarrow q$, but $\phi(a,p) \wedge \neg\phi(a,q)$, i.e. not $\phi(a,p) \leftrightarrow \phi(a,q)$. But then even most elementary "laws" such as, e.g.,

(CLOS1) $\phi(a,p\wedge q) \leftrightarrow \phi(a,q\wedge p)$

(CLOS2) $\phi(a,p\vee p) \leftrightarrow \phi(a,p)$

or

(CLOS3) $\phi(a,p) \leftrightarrow \phi(a,\neg\neg p)$

would not be valid, and one could hardly find *any* epistemic logical law which adequately describes the factual knowledge- or belief-system of an arbitrary subject, a.

However, this sceptical conclusion rests on a very narrow conception of our everyday's attribution of propositional attitudes. When in the preceding paragraph the possibility was granted that a person *a* might not "see" that two logically equivalent propositions *p* and *q* are in fact logically equivalent, the ascription of $\phi(a,p)$ and the non-ascription of $\phi(a,q)$ will usually be based on *a*'s *verbal behaviour*. When asked whether (she believes that) *p* is true, *a* answers in the affirmative, while when asked whether (she believes that) *q*, *a* happens to answer in the negative. Now, even if one assumes that the answers were intended quite sincerely, there remain several sources for a possible clash between what *a said* and what she really *believed*. She may have misunderstood one or the other question; one of the answers may be the result of a slip of tongue; etc. In any case, the very fact that *p* and *q* are logically equivalent and hence "mean the same thing" strongly suggests that *a* did not fully *understand* the meaning of *p* and/or *q*.

In everyday's discourse, however, we standardly presuppose that the people with which we talk have an adequate understanding of what is said. Therefore we assume that their belief- or knowledge-systems satisfy certain conditions of *rationality*, in particular a certain amount of logical consistency and deductive closure.[i] In this sense one may consider the task of epistemic logic to consist (1) in elaborating the "logical" laws which one may *rationally expect* the belief- and knowledge-system of a subject *a* to obey and (2) in clarifying the analytical relations that exist between these epistemic attitudes. In the following section the former laws will be presented by sets of axiomatic principles **B1-B7**, **C1-C11**, and **K1-K8** (for the logic of Belief, Conviction, and Knowledge, respectively), while the epistemic laws interrelating these notions will be denoted as **E1-E12**. A more systematic exposition of the syntax and semantics of corresponding formal calculi may be found in Lenzen 1980a.

1. The logic of belief

In the vast majority of publications on epistemic logic it is tacitly presupposed that only one unique concept of belief has to be investigated. However, as was first argued in Lenzen 1978, at least two different concepts of belief – which display a quite distinct logical behaviour – must be carefully distinguished: “strong” and “weak” belief.

1.1 The logic of “strong belief”

Let ‘$C(a,p)$’ abbreviate the fact that person *a* is firmly convinced that *p*, i.e. that *a* considers the proposition *p* (or, equivalently, the state of affairs expressed by that proposition) as absolutely certain; in other words, *p* has maximal likelihood or probability for *a*. Using ‘Prob’ as a symbol for subjective probability functions, this idea can be formalized by the requirement:

(PROB-C) $C(a,p) \leftrightarrow \text{Prob}(a,p)=1.$

Within the framework of standard possible-worlds semantics $<I,R,V>$, $C(a,p)$ would have to be interpreted by the following condition:

(POSS-C) $V(i,C(a,p))=\mathbf{t} \leftrightarrow \forall j(iRj \rightarrow V(j,p)=\mathbf{t}).$

Here *I* is a non-empty set of (indices of) possible worlds; *R* is a binary relation on *I* such that *iRj* holds if and only if (or, for short, iff) in world *i*, *a* considers world *j* as possible; *V* is a valuation-function assigning to each proposition *p* relative to each world *i* a truth-value $V(i,p)\in\{\mathbf{t},\mathbf{f}\}$. Thus $C(a,p)$ is true (in world $i\in I$) iff *p* itself is true in every possible world *j* which is considered by *a* as possible (relative to *i*).

The probabilistic definition **POSS-C** together with some elementary theorems of the theory of subjective probability immediately entails the validity of the subsequent laws of conjunction and non-contradiction. If *a* is convinced both of *p* and of *q*, then *a* must also be convinced that *p* and *q*:

(C1) $C(a,p) \land C(a,q) \rightarrow C(a,p\land q)$.

For if both Prob(a,p) and Prob(a,q) are equal to 1, then it follows that Prob($a,p\land q$)=1, too. Furthermore, if a is convinced that p (is true), a cannot be convinced that $\neg p$, i.e. that p is false:

(C2) $C(a,p) \rightarrow \neg C(a,\neg p)$.

For if Prob(a,p)=1, then Prob($a,\neg p$)=0, and hence Prob($a,\neg p$)$\neq$1. Just like the alethic modal operators of possibility, $\Diamond$, and necessity, $\Box$, are linked by the relation $\Diamond p \leftrightarrow \neg\Box\neg p$, so also the doxastic modalities of thinking p to be possible – formally: $P(a,p)$ – and of being convinced that p satisfy the relation

(Def. P) $P(a,p) \leftrightarrow \neg C(a,\neg p)$.

Thus, from the probabilistic point of view, $P(a,p)$ holds iff a assigns to the proposition p (or to the event expressed by that proposition) a likelihood greater than 0:

(PROB-P) $V(P(a,p))=\mathbf{t} \leftrightarrow \text{Prob}(a,p)>0$.

Within the framework of posssible-worlds semantics, one obtains the following condition:

(POSS-P) $V(i,P(a,p))=\mathbf{t} \leftrightarrow \exists j(iRj \land V(j,p)=\mathbf{t})$,

according to which $P(a,p)$ is true in world i iff there is at least one possible world j – i.e. a world j which a considers as possible relative to i – in which p is true.

In view of **Def P**, the former principle of consistency, **C2**, can be paraphrased by saying that whenever a is firmly convinced that p, a will a fortiori consider p as possible. However, considering p as possible does not conversely entail being convinced that p. In general there will be many propositions p such that a considers both p and $\neg p$ as possible. Such a situation, where $P(a,p) \land P(a,\neg p)$, makes clear that unlike the operator C, P will not in general satisfy a principle of conjunction analogous to **C1**. However, the converse entailment

(C3) $P(a,p\land q) \rightarrow P(a,p) \land P(a,q)$

and its counterpart

(C4) $C(a,p{\wedge}q) \rightarrow C(a,p) \wedge C(a,q)$

clearly are valid, because the probabilities of the single propositions p or q always are at least as high as the probability of the conjunction $(p{\wedge}q)$. Similarly, since the probability of a disjunction $(p{\vee}q)$ is always at least as high as the probabilities of the single disjuncts p and q, it follows that both operators C and P satisfy a corresponding principle of disjunction:

(C5) $C(a,p) \vee C(a,q) \rightarrow C(a,p{\vee}q)$

(C6) $P(a,p) \vee P(a,q) \rightarrow P(a,p{\vee}q)$.

Now the probabilistic "proofs" of such principles are not without problems. Since its early foundations by de Finetti 1964, the theory of subjective probability has always been formulated in terms of *events*, while in the framework of philosophical logic attitudes like $C(a,p)$ are traditionally formulated in terms of *sentences*. So if one wants to apply the laws of the theory of subjective probability to the fields of cognitive attitudes, one has to presuppose (i) that for every event X there corresponds exactly one proposition p, and (ii) that the cognitive attitudes really are "propositional" attitudes in the sense that their truth is independent of the specific linguistic representation of the event X. That is, whenever two sentences p and q are logically equivalent and thus describe one and the same event X, then $C(a,p)$ holds iff $C(a,q)$ holds as well. This requirement can be formalized by the following rule:

(C7) $p \leftrightarrow q \ \vdash C(a,p) \leftrightarrow C(a,q)$.

This principle further entails that everybody must be convinced of everything that logically follows from his own convictions:

(C8) $p \rightarrow q \ \vdash C(a,p) \rightarrow C(a,q)$.

For if p logically implies q, then p is logically equivalent to $p \wedge q$; thus $C(a,p)$ entails $C(a,p \wedge q)$ (by **C7**) which in turn entails $C(a,q)$ by **C4**.

As was already stressed in section 0.2 above, there has been a long discussion whether and to which extent the epistemic attitudes of real subjects are *deductively closed*. In view of man's almost unlimited fallibility in matters of logic, some authors have come to argue that **C8** should be restricted to very elementary instances like **C4** or **C5** or to some other so-called 'surface tautologies' (cf., e.g., Hintikka 1970a). Which option one favours will strongly depend on the methodological role that one wants to assign to epistemic logic. If epistemic logic is conceived of as a *descriptive system* of people's factual beliefs, then not even the validity of the most elementary principles like **C4** seems warranted. If, on the other hand, epistemic logic is viewed as a *normative* system of *rational* belief, then even the strong condition of full deductive closure, **C8**, appears perfectly acceptable. Incidentally, if one presupposes that everybody has at least one conviction – an assumption which is logically guaranteed by some of the subsequent iteration-principles[ii] – **C8** entails the further rule

(C9) $\quad p \vdash C(a,p),$

according to which everybody is convinced of every tautological proposition (or state of affairs) p.

To round off our exposition of the logic of conviction, let us consider some laws for *iterated* epistemic attitudes. According to the thesis of the "privileged access" to our own mental states, whenever some person a is convinced of p, a knows that she has this conviction. Similarly, if a is not convinced that p, i.e. if she considers p as possible, then again she knows that she considers p as possible:

(E1) $\quad C(a,p) \rightarrow K(a,C(a,p))$

(E2) $\quad \neg C(a,p) \rightarrow K(a,\neg C(a,p)).$

Here '$K(a,q)$' abbreviates the fact that *a knows* that q. Now, clearly a knows that q only if in particular a is convinced that q:

(E3) $K(a,p) \rightarrow C(a,p)$.

Hence one immediately obtains the following purely doxastic iteration-principles

(C10) $C(a,p) \rightarrow C(a,C(a,p))$

(C11) $\neg C(a,p) \rightarrow C(a,\neg C(a,p))$.

It is easy to verify that the implications **C10** and **C11** may be strengthened into equivalences. Generally speaking, iterated doxastic operators or "modalities" are always reduceable to simple "modalities" of the types $C(a,p)$ and $\neg C(a,q)$, where p and q contain no further doxastic expressions. As a matter of fact, iterated doxastic *propositions* of arbitrary complexity can be reduced to simple, non-iterated propositions. In the end, then, the logic of conviction turns out to be structurally isomorphic to the "deontic" calculus **DE4** of Lemmon 1977 which differs from the better-known alethic calculus **S5** only in that it does not contain the "truth-axiom" $\Box p \rightarrow p$. Given the intended doxastic interpretation of "necessity" as *subjective* necessity or certainty, the failure of $C(a,p) \rightarrow p$ comes as no surprise. After all, humans are not infallible; therefore someone's conviction that p – however firm it may be – can never logically guarantee that p is in fact the case.

1.2 The logic of weak belief

While the concept of conviction, $C(a,p)$, has been defined above to obtain iff person a is absolutely *certain* that p, the more general concept of "weak" *belief*, $B(a,p)$, will be satisfied by the much more liberal requirement that person a only considers p as *likely* or as *probable*. Here the lower bound of (subjective) probability may reasonably be taken to be .5. In other words, person a believes that p iff she considers p as more likely than not:

(PROB-B) $B(a,p) \leftrightarrow \mathrm{Prob}(a,p)>1/2$.

This "weak" notion of belief also satisfies the principle of non-contradiction analogous to **C2**:

(B1) $B(a,p) \rightarrow \neg B(a,\neg p)$.

Clearly, if p has a probability greater than 1/2, then $\neg p$ must have a probability less than 1/2. On the other hand, $B(a,p)$ does not satisfy the counterpart of conjunction principle **C1**, because even if two single propositions p and q both have a probability > .5, it may well happen that $\mathrm{Prob}(a,p\wedge q)$ is < .5. For instance, let an urn contain two black balls and one white ball where one of the black balls is made of metal while the white ball and the other black ball is made of wood. Now if just one ball is drawn from the urn at random, the probability of p = 'The ball is black' equals 2/3 and is thus > 1/2; also the probability of q = 'The ball is made of wood' is 2/3 > 1/2. But the probability of the joint proposition $(p\wedge q)$ = 'The ball is made of wood and is black' only is 1/3.

It follows from the theory of probability that conjunctivity of belief is warranted only in the special case where one of the two propositions is *certain*:

(E4) $B(a,p) \wedge C(a,q) \rightarrow B(a,p\wedge q)$.

Here certainty may be said to represent a special instance of belief in the sense of:

(E5) $C(a,p) \rightarrow B(a,p)$.

The validity of this principle derives from the fact that each proposition p with maximal probability 1 *a fortiori* has a probability greater than .5! Thus, *semantically* speaking, *a*'s believing that p is entirely compatible with *a*'s being absolutely certain that p, although from a *pragmatic* point of view when person *a* says 'I believe that *p*', she thereby expresses that she is *not convinced* that *p*.[iii]

The epistemological thesis of the privileged access to (or the privileged knowledge of) our own mental states mentioned earlier in connection with principles **E1** and **E2** evidently applies not only to the particular doxastic attitude $C(a,p)$, but to the more general notion $B(a,p)$ as well. Thus, whenever person a believes that p, a knows that she believes that p; and, conversely, if she does not believe that p, she knows that she does not believe that p:

(E6) $B(a,p) \to K(a,B(a,p))$

(E7) $\neg B(a,p) \to K(a,\neg B(a,p))$.

In view of **E3** and **E5,** one immediately obtains the following "pure" iteration-laws:

(B2) $B(a,p) \to B(a,B(a,p))$

(B3) $\neg B(a,p) \to B(a,\neg B(a,p))$.

Furthermore the rules of deductive closure of belief:

(B4) $p \leftrightarrow q \ \vdash B(a,p) \leftrightarrow B(a,q)$

(B5) $p \to q \ \vdash B(a,p) \to B(a,q)$

(B6) $p \ \vdash B(a,p)$

can be justified in strictly the same way as the corresponding principles for conviction.

In order to obtain a complete axiomatization of the logic of "weak" belief, one has to introduce the somewhat unfamiliar relation of "strict implication" between *sets of propositions* $\{p_1,...,p_n\}$ and $\{q_1,...,q_n\}$ ($n \geq 2$). Let this generalization of the ordinary relation of logical implication be symbolized by $\{p_1,...,p_n\} \Rightarrow \{q_1,...,q_n\}$. This relation has been defined by Segerberg 1971 to hold iff, for logical reasons, at least as many propositions from the set $\{q_1,...,q_n\}$ must be true as there are true propositions in the set $\{p_1,...,p_n\}$. Now, just like the logical implication between p and q guarantees that the probability of q is at least as great as the probability of p, so also the strict implication between $\{p_1,...,p_n\}$ and $\{q_1,...,q_n\}$ entails that the *sum* of the probabilities of the q_1 is at least as great as the corresponding sum $\sum_{i \leq n}$ Prob(a, p_i). Therefore, if at least one proposition from $\{p_1,...,p_n\}$ is believed by a to be true (and hence has a probability > .5) and if all the other p_i are not believed by *a* to be false (and hence have a probability ≥ .5), so that in sum $\sum_{i\leq n}$ Prob(a, p_i) > n•1/2, it follows that also $\sum_{i\leq n}$ Prob(a,q_i) > n/2, and thus at least one of the q_i must be believed by *a* to be true:

(B7) $\{p_1,\ldots,p_n\} \Rightarrow \{q_1,\ldots,q_n\} \vdash B(a,p_1)\wedge\neg B(a,\neg p_2)\wedge\ldots\wedge\neg B(a,\neg p_n) \rightarrow B(a,q_1)\vee\ldots\vee B(a,q_n)$.

2. The logic of knowledge

2.1 In search of a "definition" of knowledge

Although *a*'s firm belief that *p* is true is logically compatible with *p*'s actually being false, it is a truism since Plato's early epistemological investigations in the *Theaitetos* that *a* cannot *know* that *p* unless *p* is in fact true. This first, "objective" condition of knowledge can be formalized as:

(K1) $K(a,p) \rightarrow p$.

Another "subjective" condition of knowledge has already been stated in the preceding section: **E3** says that person *a* cannot know that *p* unless she is convinced that *p*. This is a refinement of Plato's insight that knowledge requires belief – *viz*., belief of the strongest form possible. Plato had discussed yet a third condition of knowledge which is somewhat harder to grasp. In order to constitute an item of *knowledge*, *a*'s true belief must be "justified" or "well-founded". One might think of explicating this requirement by postulating the existence of certain propositions $q_1,\ldots,q_n$ which "justify" *a*'s belief that *p* by logically entailing *p*. But which epistemological status should be accorded to these "justifying" propositions? If it were only required that the q_i must all be true and that *a* is convinced of their truth, then the "third" condition of knowledge would become redundant and each true belief would by itself be "justified".[iv] On the other hand one cannot require that the q_i are *known* by *a* to be true, because then Plato's definition of knowledge as "justified" true belief would become circular.[v]

For the present purpose of investigating the *logic* of knowledge, two alternatives offer themselves. Either one treats 'knowledge' as a primitive, undefinable notion which is only partially characterized by the necessary conditions **K1** and **E3**. Or one takes the conjunction of these two conditions as already *sufficient* for *a*'s knowing that *p* – an option favoured by Kutschera 1982 and, more recently, by Sartwell 1991.[vi] Let us refer to this simple concept of knowledge as 'knowledge*' or 'K^*'. If one thus defines:

(Def. K*) $K^*(a,p) \leftrightarrow C(a,p) \wedge p$,

then the logic of knowledge* can easily be derived from the logic of conviction. This will be briefly carried out in section 2.2. The logic of a more demanding primitive notion of knowledge, $K(a,p)$, will afterwards be investigated in section 2.3.

2.2 The logic of knowledge as true, strong belief*

The first basic principle

(K*1) $K^*(a,p) \rightarrow p$

is an immediate corollary of Def. K*. Furthermore, the former conjunction-principle **C1** for strong belief directly entails a corresponding principle for knowledge*,

(K*2) $K^*(a,p) \wedge K^*(a,q) \rightarrow K^*(a,p \wedge q)$,

and the rules of deductive closure of conviction, **C7** - **C9**, analogously entail the following rules for K^*

(K*3) $p \leftrightarrow q \vdash K^*(a,p) \leftrightarrow K^*(a,q)$

(K*4) $p \rightarrow q \vdash K^*(a,p) \rightarrow K^*(a,q)$

(K*5) $p \vdash K^*(a,p)$.

It is easy to verify that **Def. K*** together with **C10** entails the iteration law

(K*6) $K^*(a,p) \rightarrow K^*(a,K^*(a,p))$.

As regards the "converse" iteration principle $\neg K^*(a,p) \rightarrow K^*(a,\neg K^*(a,p))$, two subcases must be distinguished. If *a*'s failure to know that *p* is due to *a*'s not sufficiently *believing* that *p*, then the conclusion $K^*(a,\neg K^*(a,p)$ is warranted; for in view of **C11** also

(E8) $\neg C(a,p) \rightarrow K^*(a,\neg C(a,p))$

becomes provable. If, however, $\neg K^*(a,p)$ results from a failure of the "objective" condition of knowledge*, i.e. if *p* is false although *a* is strongly convinced that *p*, then *a* will evidently *not* know that she does not know that p.[vii] Hence the logic of *K** is at least as strong as the well-known modal system **S4** but definitely weaker than **S5**. A closer characterization will be given towards the end of the next section.

2.3 The logic of a more demanding concept of knowledge

The basic principle **K1** was already dealt with in section 2.1. Second, in analogy to **K*2**, also the more sophisticated concept of knowledge along Platonian lines should be taken to satisfy the principle of conjunctivity:

(K2) $K(a,p) \wedge K(a,q) \rightarrow K(a,p\wedge q)$.

For if one assumes that *a*'s single convictions that *p* and that *q* are justified, then *a*'s combined conviction that $(p\wedge q)$ would be justified as well. Third, the methodological position outlined in the introduction of this paper validate the following rules of deductive closure also for the more ambitious concept *K*:

(K3) $p \leftrightarrow q \vdash K(a,p) \leftrightarrow K(a,q)$

(K4) $p \rightarrow q \vdash K(a,p) \rightarrow K(a,q)$

(K5) $p \vdash K(a,p)$.

Since epistemic logic is here taken as a normative theory of *rational* (or "implicit") attitudes, these rules are just as acceptable as their doxastic counterparts **C7 – C9** plus their corrolaries **K*3 – K*5**.

The *K*-analogue of the iteration law **K*6**, i.e. so-called "KK-thesis", says that whenever a person *a* knows that *p*, *a* knows that she knows that *p*:

(K6) $K(a,p) \rightarrow K(a,K(a,p))$.

In the literature surveyed in Lenzen 1978, several "counter-examples" have been constructed to show that a person *a* may know something without knowing that she knows. For instance, assume that during an examination student *a* answers the question in which year Leibniz was born by replying "In 1646". The very fact that *a* managed to give the correct answer usually is taken as sufficient evidence to conclude that *a knew the correct answer*. But *a* may *not have known* at all *that she knew* the correct answer; in fact she may have thought she was just *guessing*.

Such examples typically play on the ambiguity of the English verb 'to know' which has the meaning both of the German 'wissen' and of 'kennen'. In the former case, 'to know' is followed by a that-clause and then expresses a *propositional attitude*; while in the latter case, 'to know' is part of a direct object construction ('to know the answer'; 'to know the way'; 'to know the city of London'; etc) and then expresses no such attitude. Therefore the above "counter-example" fails to refute **K6** since *a*'s "knowing" the correct answer, i.e. her knowing the year in which Leibniz was born, does not represent a propositional attitude as would be required by **K6**. According to the premises of the story, *a* did *not* know *that* Leibniz was born in 1646 because she was not at all certain of the date. If someone really *knows* that Leibniz was born in 1646, i.e., by **E3**, if *a* is *a fortiori* convinced that Leibniz was born in 1646, then *a* can never believe that he does not know that Leibniz was born in 1646.

The argument contained in the preceding passage contains an application of another important principle which establishes an epistemic logical connection between all the three basic notions of knowledge, belief, and conviction. In its general form, it would have to be put as follows: Whenever person *a* is *convinced* that *p*, she will *believe* that she *knows* that *p*:

(E9) $C(a,p) \rightarrow B(a,K(a,p))$.

In view of certain iteration laws discussed earlier in this paper, **E9** can be strengthened into the statement that when *a* is convinced that *p*, she must be convinced that she knows that *p*.

(E10) $C(a,p) \rightarrow C(a,K(a,p))$.

Incidentally, the implications **E9** and **E10** might further be strengthened into equivalences, and because of **C10** also the following law becomes provable:

(E11) $C(a,C(a,p)) \leftrightarrow C(a,K(a,p))$.

E11 shows that knowledge and conviction are *subjectively indiscriminable* in the sense that person *a* cannot tell apart whether she is "only" convinced that *p* or whether she really knows that *p*. This observation does not remove, however, the *objective* difference between *a*'s being convinced that *p* and *a*'s knowing that *p*; only the latter but not the former attitude entails the truth of *p*. Therefore it is always ("objectively") possible that *a* is convinced of something which as a matter of fact is not true; but person *a* herself can never think this to be possible.[viii]

Because of the objective possibility of $C(a,p)\wedge\neg p$, the *K*-analogue of the doxastic iteration principle **C11**, i.e. $\neg K(a,p) \rightarrow K(a,\neg K(a,p))$, fails to hold. From the assumption that person *a* *does not* know that *p* one cannot infer that she *knows* that she does not know that *p*. For if *a* *mistakenly* believes that she knows that *p*, i.e. if $C(a,p) \wedge \neg p$, one has $\neg K(a,p)$ (because of **K1)** and yet *a* does not know of her mistake, because in view of **E9** *a* believes that she *does* know that *p*; hence she is far from believing (or even knowing) that she does *not* know that *p*.

Summing up, then, no matter whether 'knowledge' is taken in the simply sense of K^* or in the more demanding sense of K, the logic of knowledge is (isomorphic to a modal calculus) at least as strong as **S4** but weaker than **S5**. Now there is a very large – indeed, as shown in Fine 1974, an *infinite* – variety of modal systems "between" **S4** and **S5**. E.g., so-called system **S4.2** is characterized by an axiom which (when the alethic operator $\Box$ is interpreted as 'knowledge') takes the form:

(K7) $\neg K(a,\neg K(a,p)) \rightarrow K(a,\neg K(a,\neg K(a,p)))$.

Another calculus **S4.4** is axiomatized by (the $\Box$-counterpart of):

(K8) $p \wedge \neg K(a,\neg K(a,p)) \rightarrow K(a,p))$.

However, the meaning of these principles is not at all evident because common sense says little or nothing about the epistemic counterpart of the alethic modality $\Diamond\Box p$, i.e. $\neg K(a,\neg K(a,p))$. Fortunately, the laws of epistemic logic developed earlier in this paper give us a clue how to understand this complex term. It is easy to prove that person a is convinced that p iff she does not know that she does not know that p:

(E12) $\neg K(a,\neg K(a,p)) \leftrightarrow C(a,p)$.

One the one hand, $C(a,p)$ entails $C(a,K(a,p))$ (by **E10**) and a fortiori $\neg C(a,\neg K(a,p))$ (by **C2**) which in turn entails $\neg K(a,\neg K(a,p))$ by **E3**; on the other hand $\neg C(a,p)$ implies $K(a,\neg C(a,p))$ (by **E2**) and hence also $K(a,\neg K(a,p))$ by the rule **K4** in conjunction with **E3**.

In view of **E12**, then, the **S4.2**-like principle **K7** amounts to saying that when person a is convinced that p, she knows that she is convinced that p – this is exactly the content of our earlier principle **E1**. Similarly, **S4.4**-like principle **K8** states that when p is true and when a is convinced that p, then a already knows that p.

As the reader may easily verify, on the basis of **Def. K*** both

(K*7) $\neg K^*(a,\neg K^*(a,p)) \rightarrow K^*(a,\neg K^*(a,\neg K^*(a,p)))$

and

(K*8) $p \wedge \neg K^*(a,\neg K^*(a,p)) \rightarrow K^*(a,p))$

become theorems of the logic of strong belief. Hence the logic of K^* actually is (isomorphic to) **S4.4**. As regards the logical structure of the more demanding concept of knowledge, K, all that can be asserted here is that it is (isomorphic to an alethic modal system) *at least as strong as* **S4.2** *but weaker than* **S4.4**.[ix]

To conclude our discussion of the propositional logic of knowledge, let it just be pointed out that a possible-worlds semantics for K can be given along the following lines:

(POSS-K) $V(i,K(a,p))=\mathbf{t} \leftrightarrow \forall j(iRj \rightarrow V(j,p)=\mathbf{t})$.

Here 'R' denotes an accessibility relation between worlds which obtains iff world j is compatible with (or "possible" according to) all that a knows in world i.

3. "Quantifying in" and other problems in first order epistemic logic

During the late 50ies and 60ies a large controversy concerning the very possibility of quantified modal logic took place among such prominent philosophers as, e.g., W.V. Quine, J. Hintikka, and D. Kaplan. In what follows, only the most fundamental issues will be touched while the historical development of the discussion must remain out of consideration.[x] The main source of the problem of "quantifying in" is the failure of substitutivity of co-referential singular terms within modal contexts:

3.1 Referential opacity

According to a by now familiar terminology, a context ϕ is said to be *referentially transparent* with respect to a term t iff t may be replaced in ϕ, *salva veritate*, by any coreferential term t':

(SUB-ϕ) $\forall tt'(t= t' \rightarrow (\phi(t) \leftrightarrow \phi(t')))$.

If **SUB-ϕ** does not hold, ϕ is said to be *referentially opaque*. Now, epistemic operators such as $B(a,p)$, $C(a,p)$, or $K(a,p)$ evidently generate referentially opaque contexts. For example, in Sophocles' famous drama, although Iocaste was (identical with) Oedipus' mother – $i = \iota xM(x,o)$ – the fact that Oedipus knew he was in love with Iocaste did not at all entail that Oedipus knew he was in love with his mother, i.e., making use of some straightforward abbreviations, one has $K(o,L(o,i))$ but $\neg K(o,L(o,\iota xM(x,o)))$. In general, the inference from $K(a,\phi(t))$ to $K(a,\phi(t'))$ seems warranted only if, instead of the mere identity $t=t'$, one has the stronger premise that this identity is *known* by subject a to hold:

(SUB1) $\forall tt'(K(a,t=t') \rightarrow (K(a,\phi(t)) \leftrightarrow K(a,\phi(t'))))$.

In the case of the other epistemic operators $C(a,p)$ and $B(a,p)$, one obtains analogously:

(SUB2) $\forall tt'(C(a,t=t') \rightarrow (C(a,\phi(t)) \leftrightarrow C(a,\phi(t'))))$

(SUB3) $\forall tt'(C(a,t=t') \rightarrow (B(a,\phi(t)) \leftrightarrow B(a,\phi(t'))))$. [xi]

Now, the referential opacity of epistemic contexts appears to render any quantification into these contexts dubious. Consider, e.g., the elementary law of existential generalization:

(EX1) $\phi(t) \rightarrow \exists x\phi(x)$,

and let ϕ be some epistemic statement such as, e.g., 'Oedipus believes that his mother is dead', $B(o,D(\iota xM(x,o)))$. Because of his ignorance concerning the identity of Iocaste and his mother, $\neg K(o,i=\iota xM(x,o))$, Oedipus certainly does *not* believe that *Iocaste* is dead: $\neg B(o,D(i))$. But then, one might argue, the premise $B(o,D(\iota xM(x,o)))$ does not entail the existential proposition $\exists xB(o,D(x))$ asserting that there exists someone, x, such that Oedipus believes x to be dead. For, according to Quine, $\exists x\phi(x)$ is *true* only if the open sentence $\phi(x)$ expresses a property which is true *of* some individual x, no matter which way we happen to refer to this individual. But it is evidently *not* true *of* Oedipus' mother, i.e. *of* Iocaste, that Oedipus would believe *her* to be dead since Oedipus does not believe *that* Iocaste is dead. Thus the inference from

$B(o,D(\iota xM(x,o)))$ to $\exists xB(o,D(x))$ should not be considered as logically valid (although, with respect to the particular predicate $D(x)$ chosen in our example, the truth of the conclusion $\exists xB(o,D(x))$ would most likely seem to be warranted by Oedipus' other beliefs).

Closely related to this *logical* objection is a *linguistic* objection of Quine's pertaining to the meaningfulness of quantified epistemic expressions in general. The content of someone's epistemic attitude usually is a *state of affairs* which can be expressed by some proposition p. Accordingly epistemic operators such as 'a believes that' or 'a knows that' (or, for that matter, also other modal operators such as 'it is necessarily true that') have to be followed by a *full*, "closed" sentence p, e.g. $p = F(t)$. The propositional operators "seal off" the subsequent that-clause in a way that the replacement of the singular term t by a variable x as, e.g., in 'a believes that $F(x)$' produces an syntactically ill-formed expression which, in contrast to the open sentence $F(x)$, cannot be taken to express a real *property*. For, a linguistic expression $\phi(x)$ denotes a property only if, for every individual x, ϕ either applies to – or fails to apply to – x regardless of the way in which we happen to refer to x. As was argued in connection with principles **SUB1 – SUB3**, however, in the case of epistemic expressions this condition is not fulfilled. Anyway, according to Quine, a quantified "sentence" like $\exists xB(a,F(x))$ – or its "ordinary language"-counterpart 'There exists some individual x such that a believes that x [or *it*] is F' – is devoid of a sound interpretation and hence, strictly speaking, *meaningless*. In the next section it will be shown how these objections can be overcome once an important distinction between two different kinds of epistemic expressions is taken into account:

3.2 De dicto and de re

Epistemic phrases such as 'a believes t to be F' or 'a knows t to be F' admit of two quite distinct interpretations: first, the more common *de dicto* reading where the content of a's belief or knowledge is the "dictum", i.e. the sentence or proposition, that t is F; second, a somewhat less common *de re* interpretation according to which the complex property of being believed or known by a to be F is attributed to the individual (or ”res”) t. While *de dicto* sentences can be represented by means of our standard operators in the usual manner:

$B(a,F(t))$ – a believes that t has the property F

$C(a,F(t))$ – a is convinced that t has the property F

$K(a,F(t))$ – a knows that t has the property F,

de re sentences appear to require a new formalism. Let $B(a,F)$, $C(a,F)$, and $K(a,F)$ abbreviate, for any epistemic subject a and for any ”normal” predicate F, the complex properties of being (weakly or strongly) believed or known by a to be F. Then *de re* sentences will take the following symbolic form:

$B(a,F)(t)$ – t is (weakly) believed by a to be F

$C(a,F)(t)$ – t is strongly believed by a to be F[xii]

$K(a,F)(t)$ – t is known by a to be F.

This type of formal representation – and the characterization of the epistemic predicates $B(a,F)$, $C(a,F)$, and $K(a,F)$ as expressing complex *properties* – is meant to suggest that *de re* sentences are referentially *transparent*. If some ”res” t has the property of being believed or known by a to be F, then it does not matter in which way we refer to that individual; i.e. if t' is identical with t, then t' also has this property. Thus we may assume that the following principles hold:

(SUB4) $t=t' \rightarrow (B(a,F)(t) \leftrightarrow B(a,F)(t'))$

(SUB5) $t=t' \rightarrow (C(a,F)(t) \leftrightarrow C(a,F)(t'))$

(SUB6) $t=t' \rightarrow (K(a,F)(t) \leftrightarrow K(a,F)(t'))$.

Furthermore, given the intended interpretation of our *de re* sentences, they evidently admit of existential generalization. Clearly, if t has the property of being believed or known by a to be F, then there exists some individual x which has this property:

(EX2) $B(a,F)(t) \rightarrow \exists x B(a,F)(x)$

(EX3) $C(a,F)(t) \rightarrow \exists x C(a,F)(x)$

(EX4) $K(a,F)(t) \rightarrow \exists x K(a,F)(x)$.

Similarly, if every ”res” *x* has the property of being believed or known by *a* to be *F*, then *t* must have this property, too:

(UN1) $\forall x B(a,F)(x) \rightarrow B(a,F)(t)$

(UN2) $\forall x C(a,F)(x) \rightarrow C(a,F)(t)$

(UN3) $\forall x K(a,F)(x) \rightarrow K(a,F)(t)$.

Note that all quantified expressions in **EX2- EX4** and **UN1 – UN3** are *de re* constructions which – unlike their *de dicto* counterparts discussed in the previous section – do not fall under Quine’s verdict of being ungrammatical.

Next it remains to be investigated which logical relations exist between epistemic propositions *de dicto* and *de re*. For convenience we will set aside the attitudes of weak and strong belief and concentrate instead on knowledge. Under which circumstances will it be allowed to ”export” the singular term *t* occurring within the *de dicto* construction ‘*a* knows that *t* is *F*’ so as to infer that *t* has the property of being known by *a* to be *F*, and *vice versa*? To answer these questions one first has to state precise truth conditions for knowledge-sentences *de dicto* and *de re*. Unfortunately, there is little agreement concerning the general framework within which such a semantics should best be developed. In particular it is still somewhat controversial in which sense one and the same individual *t* can be assumed to exist in (or to be identifiable across) different possible worlds. E.g., according to the ”counterpart-theory” developed in Lewis 1968, the domains of two such worlds should always be taken to be set-theoretically disjoint: If *t* exists in a certain world *i*, then not *t* himself but at best one of his ”counterparts” t^* can exist in another world $j \neq i$. In what follows, however, we will rather adopt an approach suggested by Kripke 1972 according to which a possible-worlds model $<U,I,R,V>$ should always be based on a common universe of discourse, U, i.e., for every world $i\epsilon I$, the domain of i is one and the same set U.[xiii]

Within this Kripkean framework our general condition **POSS-K** mentioned in section 2.3 immediately combines with the usual interpretation of the first-order formula $p = F(t)$ to yield the following truth condition for *de dicto* knowledge statements: $K(a,F(t))$ is true under the interpretation V in a world i iff in every world j which is "accessible" from i (i.e. which is possible according to all that a knows in i) V makes $F(t)$ true in j; and the latter condition, $V(j,F(t))=\mathbf{t}$, means more specifically that the object assigned by the interpretation V to t in world j, $V(j,t)$, belongs to the extension of the predicate F in world j, $V(j,F)$:

(POSS-K-DICTO) $V(i,K(a,F(t))) = \mathbf{t} \leftrightarrow \forall j(iRj \rightarrow V(j,t)\epsilon V(j,F))$.

Since a valuation function V can in general assign different objects x, x', x'', ... to a singular term t in different worlds i, i', i'', ..., the above truth condition amounts to the rather weak requirement that in every world j the object denoted by t in j has the property F in j. In contrast, the truth of the *de re* statement 't is known by a to be F' shall be taken to require more strictly that in each relevant world j (such that iRj) *one and the same object* x is denoted by t in j and this object x has the property F in j:

(POSS-K-RE) $V(i,K(a,F)(t)) = \mathbf{t} \leftrightarrow \exists x(\forall j(iRj \rightarrow V(j,t)=x \,\&\, x\epsilon V(j,F)))$.

According to this analysis every *de re* knowledge entails a correponding *de dicto* knowledge:

(**RE-DICTO**) $K(a,F)(t) \rightarrow K(a,F(t))$,

while the converse implication does not generally hold. In the next section we will discuss the extra premises that must be satisfied in order to infer a *de re* statement $K(a,F)(t)$ (or the existential corollary $\exists xK(a,F)(x)$) from the *de dicto* statement $K(a,F(t))$. To conclude this section let it just be mentioned that in the case of *belief* things are yet a little bit more complicated. In analogy to $K(a,F)(t)$, the truth of $C(a,F)(t)$ also does require that in each relevant world j one and the same object x is denoted by t in j and this object x has the property F in j. This furthermore warrants that for some singular term t' such that (in the actual world i) $t'=t$, $C(a,F(t'))$ will be true (in i). However, subject a may perhaps not know of this identity and may therefore fail to believe that t himself has property F: Remember Quine's famous 1956 scenario of Ralph's beliefs concerning t=”Bernard J. Ortcutt” and t'=”a certain man in a brown hat”!

3.3 Rigid designators and 'Knowing who t *is'*

When it comes to designing a formal calculus of first order epistemic logic, it seems very convenient to interpret (at least a subset of) the singular terms as *rigid designators* where t designates an object x *rigidly* iff $V(i,t)=x$ for every $i\varepsilon I$, i.e. iff t refers to one and the same individual x in each possible world. Kripke 1972 argued that the proper names of our ordinary discourse actually are used as rigid designators while other singular terms, in particular definite descriptions, do not always designate their referents in a rigid way. Without entering into the philosophical discussion of this issue here, let us simply postulate that the *names* b, b', b'', ... of our formal language are interpreted (by the respective valuation function V) as rigid designators:

(**RIGID**) $\quad V(i,b)=V(i',b)$ for all $i,i'\varepsilon I$,

while the denotation of a definite description $\iota x\phi x$ in world j, $V(j,\iota x\phi x)$, is logically determined by $V(j,\phi)$ and may thus vary from world to world. It then easily follows that the crucial inference

(**DICTO-RE**) $\quad K(a,F(b)) \rightarrow K(a,F)(b)$

and hence – in view of **EX4** – also ”quantifying in”

(**QUANT-IN1**) $K(a,F(b)) \rightarrow \exists x K(a,F)(x)$.

is valid for any rigid designator *b*.

If, however, *t* is a non-rigid singular term, then the corresponding inferences require an extra premise to guarantee that *t* refers to one and the same individual in at least all *relevant* worlds, i.e. in every $j\varepsilon I$ such that iRj. According to Hintikka 1962, this premise should be paraphrased as '*a* knows who *t* is'. Unfortunately, the truth conditions for this informal requirement are rather vague. Consider, e.g., *t* = 'the 1998 President of the United States'. What kinds of facts must a subject *a* know in order to know who the 1998 US-President is? Does it suffice that *a* just knows his *name*, or will *a* also have to know certain facts about the *person* Bill Clinton; must *a* furthermore be able to *identify* Bill Clinton under "normal" circumstances; etc.? In view of these indeterminacies one better forgets the informal reading '*a* knows who *t* is' and considers instead its formal counterpart which Hintikka represents as $\exists x K(a,x=t)$. Again, however, this condition is not without problems. As was rightly stressed by Quine, any quantified "sentence" of the type $\exists x K(a,\Phi(x))$ involving an epistemic *de dicto* operator would have to be "translated" as 'There exists some individual *x* such that *a* knows that *x* satisfies condition Φ'. But any such locution is grammatically ill-formed. The only meaningful interpretation of quantified epistemic sentences consists of the *de re* construction 'There exists some individual *x* such that *x* is known by *a* to satisfy Φ' – $\exists x K(a,\Phi)(x)$. Hence the crucial prerequisite for "quantifying in" the singular *de dicto* statement $K(a,F(t))$ has to be formalized more exactly by the condition $\exists x K(a,=t)(x)$ which says that there exists some individual *x* such that *x* is known by *a* to be (identical to) *t*:

(QUANT-IN2) $K(a,F(t)) \wedge \exists x K(a,=t)(x) \rightarrow \exists x K(a,F)(x)$.

Note, incidentally, that the only individual which may ever satisfy the condition $K(a,=t)$ is, of course, *t* itself. For, in view of the truth condition of knowledge, if some *x* is *known* by *a* to be (identical to) t, then *a fortiori* x has to be (identical to) t. Thus the crucial premise in **QUANT-IN2** might as well be formulated by requiring that t itself has the property of being known by a to be (identical to) t! Unlike the *de dicto* formula $K(a,t=t)$, the somewhat queer-looking *de re* sentence $K(a,=t)(t)$ is not trivially satisfied by arbitrary subjects a.[xiv] In view of the semantic principle **POSS-K-RE** stated in section 3.2 above, $V(i,K(a,=t)(t))=\mathbf{t}$ requires that there exists some individual x such that in every relevant world j $V(j,t)=x$ and $x\varepsilon V(j,=t)=\{V(j,t)\}$, i.e. in every world j such that iRj the singular term t has to be interpreted by V as designating one and the same individual x (*viz*, "the" t in the real word i).

Thus the equivalence $K(a,=t)(t) \leftrightarrow \exists xK(a,=t)(x)$, or also $K(a,=t)(t) \leftrightarrow \exists x(x=t \wedge K(a,=t)(x))$, turns out to be valid. More generally, just like in ordinary first order logic with identity any singular staement $\Psi(t)$ is provably equivalent to $\exists x(x=t \wedge \Psi(x))$, so also every singular *epistemic de re* sentence $K(a,\phi)(t)$ turns out to be equivalent to the quantified formula $\exists x(x=t \wedge K(a,\phi)(x))$:

(**RE-QUAN**) $\quad K(a,\phi)(t) \leftrightarrow \exists x(x=t \wedge K(a,\phi)(x))$.

This equivalence provides the basis for a possible simplification of our formalism. In order to distinguish *de re* from *de dicto* sentences, the ordinary propositional operator $K(a,p)$ had been supplemented in section 3.2 by a predicate-forming operator $K(a,\phi)$ which, for any predicate ϕ, yields the epistemic predicate 'is known by a to be ϕ'. Within the realm of *quantified* epistemic sentences, however, the *de dicto*/*de re* distinction is superfluous. As was stressed above, there is no meaningful way to formulate quantified *de dicto* sentences; every quantified epistemic sentence always has to be understood *de re*! Therefore we might for convenience retain the ordinary *de dicto* operator to formally represent quantified (de re) sentences according to the subsequent

(**CONVENTION**) $\quad \exists xK(a,\phi(x)) \leftrightarrow \exists xK(a,\phi)(x)$

$$\forall xK(a,\phi(x)) \leftrightarrow \forall xK(a,\phi)(x).$$

In particular, the condition $\exists x(x=t \wedge K(a,\phi)(x))$ as it occurs in **RE-QUAN** might be rewritten as $\exists x(x=t \wedge K(a,\phi(x))$ and we would thus obtain the following formal representation of the singular *de re* knowledge sentence 't is known by a to be F': $\exists x(x=t \wedge K(a,\phi(x))$ (and similarly for the other epistemic operator of strong and weak belief). In sum, then, we would obtain a first order calculus with only one type of epistemic operator $K(a,\phi)$, $C(a,\phi)$, and $B(a,\phi)$. These have to be interpreted *de dicto* whenever ϕ is a "closed" sentence or proposition p, but they have to be interpreted *de re* when $\phi(x)$ is a "open" sentence with the variable x being bound by a quantifier $\exists x$ or $\forall x$ outside the epistemic operator.

To conclude, let it be mentioned that the general semantic approach advocated here – i.e. the choice of possible-worlds models $<U,I,R,V>$ with a common universe of disourse for each possible world i – validates the following epistemic counterparts of the so-called "Barcan formula" and "converse Barcan formula" (of alethic modal logic):

(UN4) $B(a,\forall xF(x)) \rightarrow \forall xB(a,F)(x)$

(UN5) $C(a,\forall xF(x)) \rightarrow \forall xC(a,F)(x)$

(UN6) $K(a,\forall xF(x)) \rightarrow \forall xK(a,F)(x)$

(UN7) $\forall xC(a,F)(x) \rightarrow C(a,\forall xF(x))$

(UN8) $\forall xK(a,F)(x) \rightarrow K(a,\forall xF(x))$.

The invalidity of the B-counterpart of **UN7** is due to the fact that the operator of "weak belief" does not satisfy a conjunction principle analogous to **C1** or **K2**. In the simplified calculus based on the above **CONVENTION**, the *de re* components of the laws **UN4 – UN6** might, of course, be symbolized by means of the apparently *de re* formulae $\forall xB(a,F(x))$, $\forall xC(a,F(x))$, and $\forall xK(a,F(x))$, respectively. Yet this convenient formalization should not seduce anyone to overlook the important difference between these two kinds of propositions which corresponds to the Medieval distinction between propositions ”in sensu composito”, e.g., $K(a,\forall xF(x))$, and propositions ”in sensu diviso”, e.g., $\forall xK(a,F(x))$.

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[i] For a recent defense of this view cf., e.g., Meyer 1998.

[ii] Clearly, since $C(a,p)\vee\neg C(a,p)$ holds tautologically, **C10** and **C11** entail that $\{C(a,C(a,p)) \vee C(a,\neg C(a,p))\}$ is epistemic-logically true. So either way there exists a q such that $C(a,q)$.

[iii] Cf. Lenzen 1995 for a closer discussion of the differences between (and the dependency of) the semantics and the pragmatics of epistemic utterances.

[iv] Clearly, if $C(a,p)\wedge p$, then there exist some $q_1,\ldots,q_n$ such that the q_i are true and $C(a,q_i)$ and $\{q_1, \ldots,q_n\}$ logically entail p, *viz*., $q_1=\ldots=q_n=p$!

[v] For a closer discussion of this problem the reader is referred to part D of this Handbook, esp. to the contribution on the "Analysis of Knowledge".

[vi] Cf. for a closer discussion Beckermann 1997.

[vii] Otherwise the assumption $C(a,p)\wedge\neg p$ would entail a contradiction, i.e. $C(a,p) \rightarrow p$ would become a theorem of the logic of strong belief.

[viii] This observation not only represents the key for the resolution of several epistemic "paradoxes" but also helps to clarify the problems that prominent philosophers encountered during their epistemological reflections on the nature of knowledge and belief. For a more detailed discussion of the "surprise examination paradox" cf. Lenzen 1976. Lenzen 1980b offers an analysis of Wittgenstein's sometimes confused discussion of "Moore's paradox" in his late booklet 1962.

[ix] Cf. Lenzen 1979 for a closer discussion of further candidates for the logic of knowledge.

[x] Cf. Quine 1956, Hintikka 1961, and Kaplan 1969; Hintikka 1975 tries to summarize the controversy and he also mentions various other writers who had contributed to the discussion of "quantifying in".

[xi] In view of the non-conjunctivity of the operator $B(a,p)$, the premise $B(a,t=t')$ is too weak to warrant the inference from $B(a,\phi(t))$ to $B(a,\phi(t'))$). One here needs a stronger premise such as $C(a,t=t')$ or $K(a,t=t')$; cf. principle **E4** stated in section 1.2 above.
[xii] Interestingly, neither in English nor in German does there exist an idiomatic locution expressing such a strong *de re* belief in terms of ‚being convinced' or ‚being certain'.
[xiii] Let it be noted in passing that this does not entail that every individual "existing" in the actual world also "exists" in every other possible world (and hence "exists necessarily"). Real existence can be regarded as an empirical, contingent property which does not automatically apply to every individual in the domain of world *i*! Another position concerning the issue of "trans-world-identity" has been defended by Hintikka (1969, 1970b).
[xiv] The reason being that the first occurrence of 't' as part of the complex epistemic predicate $K(a,=t)$ is referentially *opaque*, i.e. $t=t'$ does not entail that x has property $K(a,=t)$ iff x has property $K(a,=t')$. The second occurrence of 't' in $K(a,=t)(t)$, however, is referentially transparent, i.e. $t=t'$ and $K(a,=t)(t)$ entail that $K(a,=t)(t')$.