PROBABILISTIC INTERPRETATIONS OF EPISTEMIC CONCEPTS

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Though there are intimate connections between subjective probability concepts on the one side and the concepts of belief on the other, the question whether a doxastic (or even an epistemic) logic can be built up on the basis of probability theory seems to have been widely neglected in the vast literature on epistemic logic.[1] I want here to investigate the possibility of definitions of belief and knowledge in terms of either a probability *relation* or a probability *function*.

Let us start from a propositional calculus, PC, using $\neg$, $\wedge$, $\vee$, $\supset$ and $\equiv$ as the functors negation, conjunction, disjunction, (material) implication, and equivalence, respectively, and define:

D1: A (subjective) probability relation '(for the person *a*) the event (expressed by the sentence) *p* is at most as probable as the event (…) *q*', formally: $p \leq q$ (we will drop explicit reference to *a* throughout what follows), is a two-place relation on the set of all sentences of PC such that

P1: $p = q$, i.e. $p \leq q$ and $q \leq p$, whenever $p \equiv q$ is tautological
P2: $\leq$ is a weak order, i.e. transitive and connex
P3: $k \leq p$ for all *p*, where $\neg k$ is tautological
P4: $p \vee r \leq q \vee r$ for all *p*, *q*, *r* with $p \leq q$, $p \wedge r \leq k$, and $q \wedge r \leq k$.

In terms of $\leq$ two concepts of belief can be defined in a straightforward manner:

D2: $B_w p := \neg p < p$ (i. e. it is not the case that $p \leq \neg p$)
D3: $B_s p := p = t$ (where *t* is a tautology).

According to **D2**, *a* believes (in the weak sense) that *p* iff *a* thinks *p* more probable than not-*p*; and according to **D3**, *a* believes that *p* (in the strong sense) iff *p* is "practically certain" for *a*. Let us see if these concepts satisfy the axioms of any "standard" system of doxastic logic!

First of all, unlike in other branches of modal logic, there is little agreement as to which system of doxastic logic might be called standard. Whereas, with '*N*' denoting the necessity operator, the following sets of axioms and rules have come to be regarded as standard for (alethic) modal logic:

RN: $p \models Np$
N1: $Np \supset p$ **T**
N2: $N(p \supset q) \supset (Np \supset Nq)$ **S4**
N3: $Np \supset NNp$ **S5**
N4: $\neg Np \supset N\neg Np$

almost every corresponding doxastic principle:

RB: $p \models Bp$
B1: $Bp \supset \neg B\neg p$ **BT**
B2: $B(p \supset q) \supset (Bp \supset Bq)$ **BS4**
B3: $Bp \supset BBp$ **BS5**
B4: $\neg Bp \supset B\neg Bp$

has been subject to extensive criticism. I do not want, however, to discuss these problems here.

Secondly, modal logic usually admits of iterated modalities, whereas the relation $\leq$ is defined for sentences of PC only. It is far from clear how sentences like '$(p \leq q) \leq (r \leq s)$' ('*p* being at most as probable as *q* is at most as probable as *r* being at most as probable as *s*') should be interpreted in general (let alone higher-degree sentences). But it does not seem totally implausible to expand **D1** to *certain* sentences of this form and accordingly postulate:

P5: $p \leq q$ implies $(p \leq q) = t$ and $[\neg(p \leq q)$, i.e.$]$ $q < p$ implies $(q<p) = t$.

Now, if the doxastic principles mentioned above are restricted to sentences of PC it is easy to confirm that both B_w and B_s satisfy **RB** (which by our restriction simply becomes equivalent to the axiom **BO**: Bt), **B1**, **B3**, and **B4**. **B2**, however, is not true of B_w, for **B2** + **BO** imply $Bp \wedge Bq \supset B(p \wedge q)$. But *p* being more probable than not-*p* and *q* being more probable than not-*q* evidently do not imply $p \wedge q$ being more probable than not-$(p \wedge q)$. B_w satisfies only the related weaker principle

B2*: If *p* logically implies *q*, then $Bp \supset Bq$,

whereas unmodified **B2** is true of B_s. For if both $p \supset q$ and *p* are "practically certain" in the sense of equiprobable to *t*, then *q* must be so too. This would not be true, however, in the case of "practical certainty" defined in terms of a probability function $P(q) = r$ ('(for the person *a*) the probability of the event *q* is equal to *r*') as '$P(q) \geq 1\text{-}\varepsilon$' for any ε, however small. Remember the lottery paradox! This shows that if we had based our doxastic concepts on such a relation *P* by setting

D4: $B_r q := P(q) \geq r$, for any $.5<r<1$,

then we would have arrived at basically the same results. For clearly, $B_{.5}$ and B_1 coincide with B_w and B_s, respectively, and, for any *r* between these extreme values, B_r satisfies exactly the same doxastic principles as does B_w (provided we require in addition to the usual probability axioms that $P(q) = r$ implies $P(P(q) = r) = 1$) and $P(q) \neq r$ implies $P(P(q) \neq r) = 1$).)

While B_s has thus been proved adequate as a model of either (restricted) **BT**, **BS4**, or **BS5**, we might ask whether B_r (for any $r<1$) is inadequate, since it satisfies only **B2*** instead of **B2**. I think we must not. Though the underlying principle "It is rational to believe the consequences of one's own beliefs" certainly is not meant to be confined to *logical* consequences, neither does it suffice to show that **B2** is a necessary *condition of belief*. To mention one reason only, '$\supset$' can hardly be said to represent adequately the 'if, then'-relation which we have in mind when we speak of 'consequences of our beliefs'.

Let's now turn to knowledge! Epistemic logics can be obtained from alethic ones simply by substituting '*K*' for '*N*'. Let us refer to the resulting systems as **KT**, **KS4**, and **KS5**. The axiom

K1: $Kp \supset p$

requires only true sentences to be known (to *a*). This condition – and thus the concept of knowledge as well – cannot be defined in terms of probability alone, since *p*'s being "practically certain" (for *a*) usually is not taken to imply that *p* is true. Of course, '$p=t$' also is a necessary condition for knowledge. For if we would weaken it to '$P(p) \geq 1\text{-}\varepsilon$', we would then be forced to admit that in the "lottery case",[2] where the chance that a given ticket is the winning ticket is less than ε, we knew it is not the winning ticket although for all we know *it may be* the winning ticket.

Now, if we take these two conditions to be sufficient for knowledge, i.e. if we define

D5: $Kp := (p=t) \wedge p$,

then it is easily seen that *K* satisfies (restricted) **RK**, **K1**, **K2**, and **K3** (if we adopt P5 again). **K4**, however, will not be satisfied: If *p* is "practically certain" for *a*, though indeed it is false, then *a* does not know that *p*. But from this it does not follow that *a* knows that he doesn't know that *p*, for in view of P5, this would mean that *a* knows *p* to be false. So, "probabilistic" knowledge, *K*, is only a model of **KT** (resp. **KS4**).

Let me conclude by sketching a possible application of our interpretation of belief to certain epistemological problems. For instance, R. Binkley has proposed three principles for temporally modified belief,[3] viz. $B^1 p \supset B^2 p$; $B^1 p \supset B^1 B^2 p$; and $B^1 p \supset \neg B^1 \neg B^2 p$ (here '$B^i p$' stands for '(the person *a*) believes at time t_i that *p*'), which become invalid for B_w. This is especially evident for the first one, since (subjective) probabilities use to change in the course of time. And the following simple example shows that the other two are at least strongly open to question. Consider the experiment of drawing balls from an urn *U* without replacement. Let *U* contain two black balls and one white ball, and let '*p*' be the sentence 'the last (=3rd) draw yields a black ball'. At t_1, before the first ball is drawn, we then have $P(p)=2/3$ and thus $B_w^1(p)$. For the same reason, *a* will believe that the first draw yields a black ball. So after that draw, at t_2, the (expected) probability of *p* is only 1/2. Thus it is not unreasonable to assume that *a* believes (at t_1) that, at t_2, he will no longer believe that *p*, i.e. $B_W^1 \neg B_W^2 p$. This example shows (but doesn't prove it, of course) that the only principle valid for B_W would be $B^1 p \supset \neg B^1 B^2 \neg p$.

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[1] For a review and discussion of this literature see my "Recent Work in Epistemic Logic", to appear in *American Philosophical Quarterly* (*APQ*), ca. 1976. [This article actually appeared 1978 in *Acta Philosophica Fennica*].
[2] I owe this example to G. Harman's "Knowledge, Inference, and Explanation", *APQ* 5 (1968), p.166.
[3] In *The Journal of Philosophy* 65 (1968), p. 131